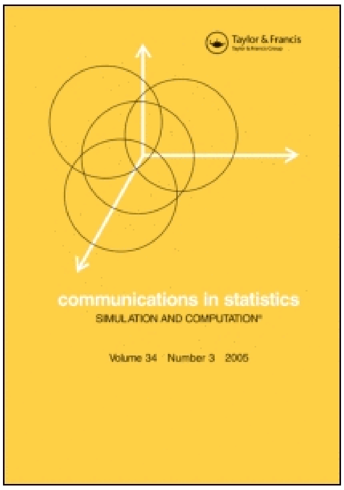

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Hai-Lin Lu [a]; Tsong-Huey Wu [b]; Hai-Wen Lu [c]
[a] Department of Management Information Science, Chia-Nan University of Pharmacy and Science, Taiwan, Republic of China [b] Department of Health Care and Administration, Chia-Nan University of Pharmacy and Science, Taiwan, Republic of China [c] Department of Electronic Commerce Management, Nanhua University, Taiwan, Republic of China

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Prediction Intervals for an Ordered Observation from Weibull Distribution Based on Censored Samples

HAI-LIN LU[1], TSONG-HUEY WU[2], AND HAI-WEN LU[3]

[1]Department of Management Information Science,
Chia-Nan University of Pharmacy and Science,
Taiwan, Republic of China
[2]Department of Health Care and Administration,
Chia-Nan University of Pharmacy and Science,
Taiwan, Republic of China
[3]Department of Electronic Commerce Management,
Nanhua University, Taiwan, Republic of China

In this article, we provide some suitable pivotal quantities for constructing prediction intervals for the jth future ordered observation from the two-parameter Weibull distribution based on censored samples. Our method is more general in the sense that it can be applied to any data scheme. We present a simulation of our method to analyze its performance. Two illustrative examples are also included. For further study, our method is easily applied to other location and scale family distributions.

Keywords Monte Carlo simulation; Order statistics; Prediction intervals; Weibull distribution.

Mathematics Subject Classification 60G25; 62N05.

1. Introduction

In most researching of reliability, the Weibull distribution is widely used as a model of lifetime data (Agresti, 1990; Bain and Engelhardt, 1991). Let us consider the two-parameter Weibull distribution with probability density function (pdf)

$$w(t) = \frac{\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1} \exp\left\{-\left(\frac{t}{\alpha}\right)^{\beta}\right\}, \quad t > 0, \tag{1}$$

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Address correspondence to Tsong-Huey Wu, Department of Health Care and Administration, Chia-Nan University of Pharmacy and Science, Tainan, Taiwan, Republic of China; E-mail: thwu@mail.chna.edu.tw

and cumulative distribution function (cdf)

$$W(t) = 1 - \exp\left\{-\left(\frac{t}{\alpha}\right)^{\beta}\right\}, \tag{2}$$

where $\beta > 0$ and $\alpha > 0$ are the shape and scale parameters, respectively. It is worth noting that if T is a random variable having the Weibull cdf given by formula (2), then the random variable $X = \ln T$ is distributed as a smallest Type I extreme value variate with pdf

$$g(x) = \frac{1}{\sigma}\exp\left(\frac{x-\mu}{\sigma}\right)\exp\left(-e^{\frac{x-\mu}{\sigma}}\right) \quad -\infty < \mu < \infty, \ \ \sigma > 0, \tag{3}$$

where $\mu = \ln\alpha$ and $\sigma = 1/\beta$. Its cdf has the form

$$G(x) = 1 - \exp\left(-e^{\frac{x-\mu}{\sigma}}\right). \tag{4}$$

In life testing studies, several lifetimes of units put on test may not be observed because of time limitations or money and material resources restrictions on data collection. Consider an experiment in which n identical components are placed on test simultaneously. Suppose the experiment was terminated when the $(n-s)$th component failed, thus censoring the last s components. Such a sample is called Type II right-censored sample. If some initial r observations are also censored, it is called Type II doubly censored.

The studies of estimating the prediction intervals of the future data are quite important and valuable in lifetime analysis. There have been several studies in the literature dealing with such problems. Kaminsky and Nelson (1998) described some interval and point prediction of order statistics. For exponential distribution, Lawless (1971) and Likeš (1974) estimated the prediction intervals based on the order statistics, $X_{(j)}$ $(r < j \le n)$, of a sample while the first r data of the sample were observed. Mann and Grubbs (1974) proposed an alternative method to construct approximate prediction intervals for a future observation. Kaminsky and Nelson (1974) constructed prediction intervals by using the best linear unbiased estimates (BLUE) of the parameters as a pivotal statistic. For the Weibull distribution, Mann and Saunders (1969) used three specially selected order statisics to predict the minimum of a single future sample. Engelhardt and Bain (1979) constructed the prediction limits for the jth smallest of some set of future observations. Fertig et al. (1980) provided Monte Carlo estimates of percentiles of the distribution of a statistics S for constructing prediction intervals of a future observation. Lawless (1973) used a conditional method to obtain a prediction interval for the first-order statistic of a set of future observations, based on previous data; Hsieh (1996) used the same technique to construct prediction intervals for future observations. Mann and Fertig (1975) constructed the tables for obtaining the best linear invariant estimates (BLIE) of parameters. Balakrishnan and Cohen (1991) proposed an approximate maximum likelihood estimates (AMLE) of parameters. All these researches are under the scheme that the available data is either right censored or doubly censored.

It is well known that the Type II censored data, the right, left, and doubly censored data are all special cases of multiple censored data. In practice, multiple Type II censored problems may arise when some components failed between two points of observation with exact times of these failure unobservable components (Balasubramanian and Balakrishnan, 1992). There are two examples in Sec. 5. In this article, we consider the general case of the multiple Type II censored data scheme. Suppose n components are placed on test in life testing. The lifetime of the first r, the middle l, and the last s components are assumed unobserved or missing. That is, we assume $X_{(r+1)} < X_{(r+2)} < \cdots < X_{(r+k)}$ (k components) and $X_{(r+k+l+1)} < X_{(r+k+l+2)} < \cdots < X_{(n-s)}$ (m components) are observable and no others.

In the next section, following the ideas of Wu et al. (2004), we present our method of constructing the prediction intervals of the future unknown observations for Type II censored data. We describe the procedure for calculating the percentiles of the distributions of the pivotal quantities, and the simulation results are compared with existing methods in Secs. 3 and 4, respectively. In Sec. 5, we illustrate our method with two examples. A brief discussion is presented in the final section.

2. Some Pivotal Quantities

Let $Y_i = \frac{X_i-\mu}{\sigma}$, then Y_i has extreme value distribution with $\mu = 0$ and $\sigma = 1$. And $Y_{(i)} = \frac{X_{(i)}-\mu}{\sigma}$ is the ith order statistic of Y. According to the general data scheme mentioned in Sec. 1, we define some pivotal quantities (general forms and proof in Appendix 1) of the following forms:

$$\widehat{U}_h = \frac{X_{(j)} - X_{(n-s)}}{\widehat{W}_h}, \quad h = 1, \ldots, 4, \;\; n-s < j \le n,$$

where for some suitably chosen function g (described in following):

$$\widehat{W}_1 = \sum_{i=r+2}^{r+k} \frac{g(E(Y_{(i)}))}{S_c}(X_{(i)} - X_{(r+1)}) + \sum_{i=r+k+l+1}^{r+k+l+m} \frac{g(E(Y_{(i)}))}{S_c}(X_{(i)} - X_{(r+1)}), \tag{5}$$

$$\begin{aligned}\widehat{W}_2 = {} & \sum_{i=1}^{r} \frac{g(E(Y_{(i)}))}{S_n}(X_{(r+2)} - X_{(r+1)}) + \sum_{i=r+2}^{r+k} \frac{g(E(Y_{(i)}))}{S_n}(X_{(i)} - X_{(r+1)}) \\ & + \sum_{i=r+k+1}^{r+k+l} \frac{g(E(Y_{(i)}))}{S_n}\left(\frac{X_{(r+k)} + X_{(r+k+l+1)} - 2X_{(r+1)}}{2}\right) \\ & + \sum_{i=r+k+l+1}^{n-s} \frac{g(E(Y_{(i)}))}{S_n}(X_{(i)} - X_{(r+1)}) \\ & + \sum_{i=n-s+1}^{n} \frac{g(E(Y_{(i)}))}{S_n}(X_{(n-s)} - X_{(r+1)}),\end{aligned} \tag{6}$$

$$\widehat{W}_3 = \left(\prod_{i=r+2}^{r+k} (X_{(i)} - X_{(r+1)})^{\frac{g(E(Y_{(i)}))}{S_c}}\right) \times \left(\prod_{i=r+k+l+1}^{r+k+l+m} (X_{(i)} - X_{(r+1)})^{\frac{g(E(Y_{(i)}))}{S_c}}\right), \tag{7}$$

$$\begin{aligned}\widehat{W}_4 = {} & \prod_{i=1}^{r} (X_{(r+2)} - X_{(r+1)})^{\frac{g(E(Y_{(i)}))}{S_n}} \times \prod_{i=r+2}^{r+k} (X_{(i)} - X_{(r+1)})^{\frac{g(E(Y_{(i)}))}{S_n}} \\ & \times \prod_{i=r+k+1}^{r+k+l} \left(\frac{X_{(r+k)} + X_{(r+k+l+1)} - 2X_{(r+1)}}{2}\right)^{\frac{g(E(Y_{(i)}))}{S_n}}\end{aligned}$$

$$\times \prod_{i=r+k+l+1}^{n-s} (X_{(i)} - X_{(r+1)})^{\frac{g(E(Y_{(i)}))}{S_n}}$$

$$\times \prod_{i=n-s+1}^{n} (X_{(n-s)} - X_{(r+1)})^{\frac{g(E(Y_{(i)}))}{S_n}}, \quad (8)$$

$$S_c = \sum_{t=r+2}^{r+k} g(E(Y_{(t)})) + \sum_{t=r+k+l+1}^{r+k+l+m} g(E(Y_{(t)})),$$

$$S_n = \sum_{t=1}^{n} g(E(Y_{(t)})).$$

$\widehat{W}_1$ and $\widehat{W}_2$ come from the ideas of arithmetic means, and $\widehat{W}_3$, and $\widehat{W}_4$ follow the concepts of geometric means. In Wu et al. (2004), $\widetilde{W}_1 = \frac{1}{n-1}\big[\sum_{i=2}^{r-1}(Y_{(i)} - Y_{(1)}) + (n-r+1)(Y_{(r)} - Y_{(1)})\big]$, i.e., Wu et al. consider the case where every datum of different position has the same weighted factor $\frac{1}{n-1}$. In the case of extreme value distribution, it seems reasonable to assume that the weight of each datum point should be different for different position. From the properties of the extreme value distribution, we suggest that the weighted factors should be equal to $g(E(Y_{(i)}))/\sum_{t=1}^{n} g(E(Y_{(t)}))$ in $\widehat{W}_h$, where $g(E(Y_{(i)}))$ is the pdf of the expected values of the ith order statistic of Y and $E(Y_{(i)})$ is the expected values of $Y_{(i)}$. The $E(Y_{(i)})$ is defined as $\int_{-\infty}^{\infty} Y_{(i)} q(Y_{(i)}) dY_{(i)}$, where $q(Y_{(i)})$ is the pdf of the ith order statistic of Y. Since the parameters μ and σ in Eqs. (5)–(8) will be cancelled (see the proof in Appendix 1), without loss of generosity, we simply treat them as standard extreme value distribution. Therefore, $E(Y_{(i)})$ will be constants and set $g(z) = e^{z}e^{-e^{z}}$, $z = E(Y_{(i)})$. It also showed that the weighted factors do not depend on the the parameters μ and σ.

For comparison, the other pivotal quantity is $\widehat{U}_a$. Let

$$\widehat{U}_a = \frac{X_{(j)} - X_{(n-s)}}{\widehat{W}_a}, \quad n-s < j \le n,$$

where

$$\widehat{W}_a = \hat{\sigma}. \quad (9)$$

The $\hat{\sigma}$ is the AMLE of σ, which can be obtained from Balakrishnan and Cohen (1991). The percentiles data of $\widehat{U}_a$ distribution are listed in Table A5 (in Appendix 2).

From Eqs. (5)–(8), the distributions of $\widehat{U}_h$ depend only on n, r, k, l, m, s, j, but not on μ and σ. Then, we have:

$$1-\alpha = P\left\{0 < \frac{X_{(j)} - X_{(n-s)}}{\widehat{W}_h} < \hat{u}_h(1-\alpha; n, r, k, l, m, s, j)\right\}$$
$$= P\big\{X_{(n-s)} < X_{(j)} < X_{(n-s)} + \hat{u}_h(1-\alpha; n, r, k, l, m, s, j)\cdot \widehat{W}_h\big\}.$$

Therefore, $(X_{(n-s)},\ X_{(n-s)} + \hat{u}_h(1-\alpha; n, r, k, l, m, s, j)\cdot\widehat{W}_h)$, $h = 1, \ldots, 4$ are one-sided $100(1-\alpha)\%$ prediction intervals of $X_{(j)}$ based on $m+k$ observations.

Table 1
The properties of estimates of parameters $\tilde{\mu}$ and $\tilde{\sigma}$ for each combination of $n, r, k, l, m, s,$ and j based on 60,000 random samples

n	r	k	l	m	s	j	$\tilde{\mu}_{mean} \pm \tilde{\mu}_{s.d.}$	$\tilde{\sigma}_{mean} \pm \tilde{\sigma}_{s.d.}$
13	0	10	0	0	3	11	−0.009147 ± 0.353776	0.921192 ± 0.344354
25	0	20	0	0	5	21	−0.001123 ± 0.228614	1.000021 ± 0.201247
30	0	24	0	0	6	25	0.000347 ± 0.208632	0.999757 ± 0.183330
35	0	28	0	0	7	29	−0.001845 ± 0.191873	1.000458 ± 0.169762
40	0	36	0	0	4	37	−0.001770 ± 0.172441	0.999497 ± 0.146308
45	0	36	0	0	4	37	−0.001070 ± 0.169018	1.000042 ± 0.149165
50	0	40	0	0	10	41	0.000579 ± 0.159355	0.999805 ± 0.141004
55	0	44	0	0	11	45	−0.000710 ± 0.151812	1.000265 ± 0.134123
60	0	54	0	0	6	55	−0.005359 ± 0.139604	1.001190 ± 0.120548

3. Calculations and Algorithm

The exact distributions of the pivotal quantities $\widehat{U}_h$ $(h = 1, \ldots, 4)$ cannot be derived algebraically, but we can approximate the distributions of $\widehat{U}_h$ $(h = 1, \ldots, 4)$ by using large quantities of the Monte Carlo sampling with some programming algorithms (in Appendix 3) to generate the percentiles of $\widehat{U}_h$. All the simulation were run with the aid Microsoft Quick Basic 4.5 program and Foxbase database software package. The procedures for generating the percentiles of $\widehat{U}_h$ are as follows.

1. Set $\mu = 0$, $\sigma = 1$. (For providing the properties of parameters $\tilde{\mu}$ and $\tilde{\sigma}$ of the random samples generated by computer, 60,000 Monte Carlo runs are done for each combination of n, r, k, l, m, s, j (some selected cases). The results are presented in Table 1. Using Table 5.3 in Mann et al. (1974) for case of $n = 13$ and Table 1 in Mann and Fertig (1975) for remaining cases to obtain the necessary weights, we can calculate their BLIE's of μ and σ, respectively. The $\tilde{\mu}_{mean}$ and $\tilde{\sigma}_{mean}$ of those random samples are very close to 0 and 1, respectively.)
2. Calculate the following statistics: $\widehat{U}_1$ in (5), $\widehat{U}_2$ in (6), $\widehat{U}_3$ in (7), $\widehat{U}_4$ in (8).
3. In Steps 1 and 2, 600,000 replicates are used to compute the percentiles of $\widehat{U}_h$ $(h = 1, \ldots, 4)$ for each combination of n, r, k, l, m, s, j.
4. Sort 600,000 results of each combination of n, r, k, l, m, s, j in ascending order.
5. Retrieve the value of $\widehat{U}_h$ $(h = 1, \ldots, 4)$ under different significance levels of α.

From the above procedures, we obtain the values of $\widehat{U}_h$ $(h = 1, \ldots, 4)$ according to the exact position of $\widehat{U}_h$ $(h = 1, \ldots, 4)$ in Step 4.

In our simulation, the percentiles for any combination of $n, r, k, l, m, s,$ and j are easily computed. To save space, we only list part of the percentiles of $\widehat{U}_h$ $(h = 1, \ldots, 4)$ in Tables A1–A4 (in Appendix 2).

4. Comparison

In this section, we compare the performance of our method with U_a in Eq. (9). We calculate their average lengths of 95% prediction intervals, and coverage probabilities for some selected combinations of n, r, k, l, m, s, j. Referring to the data scheme mentioned in Sec. 1, the simulation is computed by the following procedures.

Table 2
The average length of 95% prediction intervals, and coverage probabilities for $X_{(j)}$ by different statistics: $\mu = 0$, $\sigma = 1$

n	r	k	l	m	s	j	$\widehat{U}_1$	$\widehat{U}_2$	$\widehat{U}_3$	$\widehat{U}_4$	$\widehat{U}_a$
The average length of prediction interval (coverage probability)											
10	0	3	0	0	7	4	3.034146	2.834668	3.554925	2.868328	2.729171
							(95.23%)	(95.11%)	(95.22%)	(95.18%)	(95.01%)
10	2	5	0	0	3	8	1.261808	1.208341	1.462240	1.341985	1.174512
							(95.04%)	(94.97%)	(95.34%)	(95.18%)	(94.92%)
10	1	3	1	2	3	8	1.156878	1.123969	1.335747	1.196265	1.094455
							(94.4%)	(94.52%)	(94.58%)	(94.46%)	(94.4%)
13	0	10	0	0	3	11	0.868976	0.846851	0.917785	0.885132	0.762706
							(95.04%)	(95.01%)	(94.93%)	(94.99%)	(95.15%)
13	3	6	0	0	4	10	0.882302	0.852831	0.999311	0.945090	0.831682
							(94.69%)	(94.86%)	(94.37%)	(94.53%)	(94.85%)
13	2	5	1	1	4	10	0.853402	0.808929	0.982991	0.869487	0.788812
							(94.49%)	(94.56%)	(94.5%)	(94.49%)	(94.58%)
20	0	10	0	0	10	11	0.573943	0.545609	0.601589	0.555719	0.500980
							(94.7%)	(94.71%)	(94.64%)	(94.66%)	(94.87%)
20	2	4	2	2	10	11	0.570727	0.544830	0.638461	0.556134	0.590064
							(95.14%)	(95.08%)	(95%)	(95.05%)	(95.22%)
40	0	30	0	0	10	31	0.248098	0.244359	0.251093	0.247359	0.224264
							(94.79%)	(94.82%)	(94.76%)	(94.78%)	(94.68%)
40	10	20	0	0	10	31	0.236455	0.233026	0.244821	0.250343	0.230777
							(94.99%)	(94.99%)	(94.86%)	(94.79%)	(95.04%)
40	8	15	3	10	4	37	0.336696	0.335886	0.349590	0.356296	0.330274
							(94.96%)	(94.94%)	(94.95%)	(94.9%)	(94.95%)
60	0	50	0	0	10	51	0.183183	0.182009	0.184992	0.183714	0.169580
							(94.73%)	(94.78%)	(94.72%)	(94.72%)	(94.87%)
60	0	40	0	0	20	41	0.153417	0.151042	0.155109	0.152380	0.139707
							(94.5%)	(94.53%)	(94.48%)	(94.48%)	(94.72%)
60	5	25	10	15	5	56	0.243745	0.243235	0.248393	0.247416	0.238034
							(95%)	(94.99%)	(95.18%)	(95.16%)	(94.77%)
80	0	60	0	0	20	61	0.119687	0.118601	0.120628	0.119432	0.110984
							(94.91%)	(94.92%)	(94.95%)	(94.97%)	(94.76%)
80	20	50	0	0	10	71	0.147838	0.147624	0.150279	0.156576	0.146158
							(94.85%)	(94.85%)	(94.76%)	(94.89%)	(94.66%)
80	10	30	20	15	5	76	0.218860	0.218164	0.223081	0.221848	0.215422
							(95.44%)	(95.43%)	(95.5%)	(95.46%)	(95.38%)
100	0	80	0	0	20	81	0.101629	0.101029	0.102296	0.101648	0.094473
							(95.46%)	(95.48%)	(95.47%)	(95.48%)	(95.55%)
100	30	60	0	0	10	91	0.132178	0.132012	0.133551	0.145260	0.131335
							(94.77%)	(94.75%)	(94.59%)	(94.49%)	(94.79%)
100	30	30	10	20	10	91	0.131747	0.131657	0.134288	0.144907	0.131126
							(94.96%)	(95.06%)	(95.01%)	(95.12%)	(95.05%)

1. Set $\mu = 0$, $\sigma = 1$.
2. Generate n ($n = 10, 13, 20, 40, 60, 80, 100$) random samples from the standard extreme value distribution.
3. Calculate the values of $\widehat{W}_h$ ($h = 1, \ldots, 4, a$), and multiply of $\widehat{W}_h$ by $\widehat{U}_h$ ($h = 1, \ldots, 4, a$) (from Tables A1–A5) for each combination of n, r, k, l, m, s, j.
4. Repeat Steps 2 and 3, execute 10,000 runs, and record all upper bounds of the confidence intervals of $X_{(n-s+1)}$ and $X_{(n-s+2)}$.
5. From the results in Steps 3 and 4, calculate the average length of the 10,000 one-sided confidence intervals $(X_{(n-s)}, X_{(n-s)} + \hat{u}_h(1-\alpha; n, r, k, l, m, s, j) \cdot \widehat{W}_h)$ and coverage probabilities for all methods.

The results of simulation are listed in Table 2. It is clear that the 95% estimated expected lengths of $\widehat{U}_2$ or $\widehat{U}_a$ are shorter. The difference among $\widehat{U}_1$, $\widehat{U}_2$, and $\widehat{U}_a$ is not significant (about 0% to 3%). It is also shown that the confidence intervals of $\widehat{U}_h$ ($h = 1, \ldots, 4, a$) have almost 95% coverage probabilities. It is interesting to note that if the sample size n is larger, then the difference of average lengths among $\widehat{U}_h$ ($h = 1, \ldots, 4, a$) will be smaller. And it also showed that simulation has the property of convergence.

5. Examples

In this section, we illustrate our method with two examples.

Example 1. Consider the following 13 components were placed on test, and the test was terminated at the time of the 10th failure (Mann and Fertig, 1973). The first ten observations are given below:

$$0.22, 0.50, 0.88, 1.00, 1.32, 1.33, 1.54, 1.76, 2.50, 3.00.$$

It is assumed that the ten observed data are from the same Weibull distribution. We transform the data to extreme value form: the logs of the ten observations are:

$$-1.541, -0.693, -0.128, 0.000, 0.278, 0.285, 0.432, 0.565, 0.916, 1.099.$$

Table 3
The one-sided 90% prediction intervals, the percentiles of $\widehat{U}_k$ ($k = 1, \ldots, 4, a$), and CMd using MLE and BLIE (Hsieh, 1996)

	$\widehat{U}_k$ (0.90; 13, 0, 10, 0, 0, 10, 11)	$\widehat{U}_k$ (0.90; 13, 0, 10, 0, 0, 10, 12)	$X_{(11)}$	$X_{(12)}$
$\widehat{U}_1$	0.2436	0.4564	(1.099, 1.5794)	(1.099, 1.9981)
$\widehat{U}_2$	0.2242	0.4184	(1.099, 1.5738)	(1.099, 1.9843)
$\widehat{U}_3$	0.2686	0.5072	(1.099, 1.6129)	(1.099, 2.0685)
$\widehat{U}_4$	0.2440	0.4580	(1.099, 1.6001)	(1.099, 2.0388)
$\widehat{U}_a$	0.6162	1.1346	(1.099, 1.5358)	(1.099, 1.9026)
(CMd, MLE)	–	–	(1.099, 5.2630)	(1.099, 8.0020)
(CMd, BLIE)	–	–	(1.099, 5.2760)	(1.099, 8.0220)

Note: CMd = Conditional Method, MLE = Maxminum Likelihood Estimates

Table 4
The one-sided 95% prediction intervals and the percentiles of $\widehat{U}_k$ $(k = 1, \ldots, 4, a)$

	$\widehat{U}_k$ (0.95; 20, 2, 4, 2, 2, 10, 11)	$\widehat{U}_k$ (0.95; 20, 2, 4, 2, 2, 10, 12)	$X_{(11)}$	$X_{(12)}$
$\widehat{U}_1$	0.1489	0.2402	(−0.45, 0.1226)	(−0.45, 0.4901)
$\widehat{U}_2$	0.1523	0.2458	(−0.45, 0.0993)	(−0.45, 0.4443)
$\widehat{U}_3$	0.1621	0.2641	(−0.45, 0.2277)	(−0.45, 0.6904)
$\widehat{U}_4$	0.1651	0.2684	(−0.45, 0.1269)	(−0.45, 0.4961)
$\widehat{U}_a$	0.5427	0.8803	(−0.45, 0.1352)	(−0.45, 0.5143)

In this case, we have $n = 13$, $r = 0$, $k = 10$, $l = 0$, $m = 0$, and $s = 3$. Applying our method to estimate the one-sided 90% prediction intervals of $X_{(11)}$ and $X_{(12)}$, the results are presented in Table 3. It is clear that the shorter prediction intervals are obtained by the pivotal quantities $\widehat{U}_2$ and $\widehat{U}_a$.

Example 2. The following are 10 observations data from Lawless (1982):

$$-3.57, -2.55, -2.02, -1.66, -1.36, -1.15, -0.95, -0.77, -0.61, -0.45.$$

It is assumed that above data were obtained from a sample of 20, which are distributed according to extreme value distribution, and the last 50% data were censored. And only from the 3rd to the 6th failure times and from the 9th to the 10th failure times are available. In other words, this is the case of $n = 20$, $r = 2$, $k = 4$, $l = 2$, $m = 2$, and $s = 10$. The one-sided 95% prediction intervals of $X_{(11)}$ and $X_{(12)}$ are listed in Table 4. It is obvious that the pivotal quantities $\widehat{U}_2$ has the shortest prediction intervals.

6. Discussion

From Table 2, the average length of prediction intervals of $\widehat{U}_1$, $\widehat{U}_2$, and $\widehat{U}_a$ are shorter than $\widehat{U}_3$ and $\widehat{U}_4$. Since $\widehat{U}_3$ and $\widehat{U}_4$ are longer than $\widehat{U}_2$, $\widehat{U}_1$ and $\widehat{U}_2$ are preferred to both of them. The average lengths of prediction intervals of $\widehat{U}_3$ and $\widehat{U}_4$ are longer than $\widehat{U}_1$ and $\widehat{U}_2$. It may be the reason that the power operations in geometric means will cause the results extended unexpectedly. Intuitively, our method produces good result because we have given different weight to each datum point. Following the algorithm of Sec. 3, it is straightforward to construct prediction intervals for the future failure time by the pivotal quantities $\widehat{U}_1$ and $\widehat{U}_2$. Note also that $\widehat{U}_1$ and $\widehat{U}_2$ can be applied to any kind of data scheme.

Comparing with the existing methods, it is true that calculation procedures of $\widehat{U}_a$ are simpler than $\widehat{U}_1$ and $\widehat{U}_2$. But since the computations of $\widehat{U}_1$ and $\widehat{U}_2$ can be easily done with computers, this seems to be not an important consideration. Furthermore, from the algorithm in Appendix 3, it is not difficult for generating and simulating larger sample size n. This makes our method to be potentially more useful than the existing ones. For further study, our simulation method will be easily applied to other location and scale family distributions.

Appendix 1

A General Form of Pivotal Quantities

The prediction intervals of our method for $X_{(j)}$ are based on a subset $\{X_{(n_i)}\}_{i=1}^{c}$ of $\{X_{(k)}\}_{k=1}^{d}$, where $1 \le n_1 < n_2 < \cdots < n_c \le d < j \le n$. Let $Y_i = \frac{X_i - \mu}{\sigma}$, then Y_i has extreme value distribution with $\mu = 0$ and $\sigma = 1$. And $Y_{(i)} = \frac{X_{(i)} - \mu}{\sigma}$ is the ith order statistic of Y. We define some pivotal quantities of the following general forms:

$$\widetilde{U}_h = \frac{X_{(j)} - X_{(n_c)}}{\widetilde{W}_h}, \quad h = 1, \ldots, 4, \;\; n_c < j \le n,$$

where

$$\widetilde{W}_1 = \sum_{i=2}^{c} \frac{g(E(Y_{(n_i)}))}{\sum_{t=2}^{c} g(E(Y_{(n_t)}))}(X_{(n_i)} - X_{(n_1)}), \tag{10}$$

$$\begin{aligned}\widetilde{W}_2 = &\sum_{i=1}^{n_1-1} \frac{g(E(Y_{(i)}))}{\sum_{t=1}^{n} g(E(Y_{(t)}))}(X_{(n_2)} - X_{(n_1)}) + \sum_{i=2}^{c} \frac{g(E(Y_{(i)}))}{\sum_{t=1}^{n} g(E(Y_{(t)}))}(X_{(n_i)} - X_{(n_1)}) \\ &+ \sum_{i=1}^{c-1} \sum_{j=n_i+1}^{n_{i+1}-1} \frac{g(E(Y_{(j)}))}{\sum_{t=1}^{n} g(E(Y_{(t)}))}\left(\frac{X_{(n_i)} + X_{(n_{i+1})} - 2X_{(n_1)}}{2}\right) \\ &+ \sum_{i=n_c+1}^{n} \frac{g(E(Y_{(i)}))}{\sum_{t=1}^{n} g(E(Y_{(t)}))}(X_{(n_c)} - X_{(n_1)}),\end{aligned} \tag{11}$$

$$\widetilde{W}_3 = \prod_{i=2}^{c} (X_{(n_i)} - X_{(n_1)})^{\frac{g(E(Y_{(n_i)}))}{\sum_{t=2}^{c} g(E(Y_{(n_t)}))}}, \tag{12}$$

$$\begin{aligned}\widetilde{W}_4 = &\left(\prod_{i=1}^{n_1-1} (X_{(n_2)} - X_{(n_1)})^{\frac{g(E(Y_{(i)}))}{\sum_{t=1}^{n} g(E(Y_{(t)}))}}\right)\left(\prod_{i=2}^{c} (X_{(n_i)} - X_{(n_1)})^{\frac{g(E(Y_{(n_i)}))}{\sum_{t=1}^{n} g(E(Y_{(n_t)}))}}\right) \\ &\times \prod_{i=1}^{c-1} \prod_{j=n_i+1}^{n_{i+1}-1} \left(\frac{X_{(n_i)} + X_{(n_{i+1})} - 2X_{(n_1)}}{2}\right)^{\frac{g(E(Y_{(j)}))}{\sum_{t=1}^{n} g(E(Y_{(t)}))}} \\ &\times \prod_{i=n_c+1}^{n} (X_{(n_c)} - X_{(n_1)})^{\frac{g(E(Y_{(i)}))}{\sum_{t=1}^{n} g(E(Y_{(t)}))}}.\end{aligned} \tag{13}$$

Theorem A.1. *If $\widetilde{W}_1$ in* (10) *is an estimator of μ and σ based on multiple Type II censored sample $X_{(n_0)} \le X_{(n_1)} \le \cdots \le X_{(n)}$ from two-parameter Weibull distribution, then $\widetilde{U}_1 = \frac{X_{(j)} - X_{(n_c)}}{\widetilde{W}_1}$ is a pivotal* (*parameter-free*) *quantity.*

Proof. Define $Y_i = \frac{X_i - \mu}{\sigma}$, then Y_i does not depend on μ and σ. And $Y_{(i)} = \frac{X_{(i)} - \mu}{\sigma}$ is the ith order statistic of Y. Let

$$Z_1 = Y_{(j)} - Y_{(n_c)}, \quad Z_2 = \sum_{i=2}^{c} \frac{g(E(Y_{(n_i)}))}{\sum_{t=2}^{c} g(E(Y_{(n_t)}))}(Y_{(n_i)} - Y_{(n_1)}),$$

where $\frac{g(E(Y_{(n_i)}))}{\sum_{t=2}^{c} g(E(Y_{(n_t)}))}$ are constants.

Define $Z_3 = Z_1/Z_2$. Both Z_1 and Z_2 are pivotal; therefore, Z_3 is also pivotal. It follows that:

$$\widetilde{U}_1 = \frac{\frac{X_{(j)}-\mu}{\sigma} - \frac{X_{(n_c)}-\mu}{\sigma}}{\sum_{i=2}^{c} \frac{g(E(Y_{(n_i)}))}{\sum_{t=2}^{c} g(E(Y_{(n_t)}))}\left(\frac{X_{(n_i)}-\mu}{\sigma} - \frac{X_{(n_1)}-\mu}{\sigma}\right)}$$

$$= \frac{Y_{(j)} - Y_{(n_c)}}{\sum_{i=2}^{c} \frac{g(E(Y_{(n_i)}))}{\sum_{t=2}^{c} g(E(Y_{(n_t)}))}(Y_{(n_i)} - Y_{(n_1)})}$$

$$= \frac{Z_1}{Z_2} = Z_3.$$

Similarly, it is easy to show that the estimators $\widetilde{U}_2$, $\widetilde{U}_3$, and $\widetilde{U}_4$ are also pivotal quantities.

Appendix 2

Table A1
The percentiles data of $\widehat{U}_1$ distribution for $X_{(j)}$

n	r	k	l	m	s	j	0.90	0.95	0.975	0.99
10	0	3	0	0	7	4	1.355785	2.201880	3.396210	5.730604
10	0	3	0	0	7	5	2.333157	3.681949	5.515064	9.268308
10	0	8	0	0	2	9	0.368497	0.503914	0.651390	0.878800
10	0	8	0	0	2	10	0.751464	0.972094	1.216881	1.570047
10	2	5	0	0	3	8	0.996771	1.426663	1.925346	2.715706
10	2	5	0	0	3	9	1.885445	2.568596	3.367626	4.607351
10	3	4	0	0	3	8	0.999649	1.437188	1.930352	2.766680
10	3	4	0	0	3	9	1.879692	2.569213	3.371813	4.671617
10	1	3	1	2	3	8	0.618781	0.864230	1.147963	1.578846
10	1	3	1	2	3	9	1.154031	1.533077	1.960601	2.666628
10	1	2	2	1	4	7	0.722657	1.030894	1.385070	1.951531
10	1	2	2	1	4	8	1.303315	1.769215	2.306443	3.195372
13	0	10	0	0	3	11	0.243662	0.330549	0.425768	0.564362
13	0	10	0	0	3	12	0.456420	0.589753	0.725857	0.931938
13	0	8	0	0	5	9	0.264716	0.369639	0.490737	0.672803
13	0	8	0	0	5	10	0.471422	0.624253	0.792476	1.040925
13	3	6	0	0	4	10	0.785583	1.106489	1.476797	2.063835
13	3	6	0	0	4	11	1.443556	1.935043	2.492976	3.334143
13	4	7	0	0	2	12	0.912021	1.239771	1.620637	2.187051
13	4	7	0	0	2	13	1.908133	2.469798	3.082472	4.043375
13	2	5	1	1	4	10	0.606973	0.843070	1.121148	1.563806
13	2	5	1	1	4	11	1.111241	1.469986	1.887287	2.478343
13	3	6	1	1	2	12	0.773738	1.050499	1.357165	1.824904
13	3	6	1	1	2	13	1.621465	2.075764	2.566789	3.304302

(continued)

Table A1
(Continued)

n	r	k	l	m	s	j	0.90	0.95	0.975	0.99
20	0	10	0	0	10	11	0.164343	0.224058	0.293705	0.396827
20	0	10	0	0	10	12	0.283892	0.371838	0.464381	0.612061
20	0	16	0	0	4	17	0.132946	0.177885	0.225080	0.290655
20	0	16	0	0	4	18	0.245659	0.309447	0.374781	0.467688
20	2	4	2	2	10	11	0.378516	0.520569	0.678417	0.920677
20	2	4	2	2	10	12	0.651078	0.854504	1.068830	1.400611
40	0	30	0	0	10	31	0.050056	0.066743	0.084676	0.108119
40	0	30	0	0	10	32	0.087866	0.109813	0.132575	0.163416
40	10	20	0	0	10	31	0.203558	0.269217	0.335965	0.428993
40	10	20	0	0	10	32	0.354067	0.442121	0.529710	0.646856
40	8	15	3	10	4	37	0.207640	0.270819	0.332608	0.416496
40	8	15	3	10	4	38	0.382493	0.469227	0.550860	0.662377
60	0	50	0	0	10	51	0.032839	0.043457	0.054636	0.069761
60	0	50	0	0	10	52	0.057801	0.071647	0.085633	0.103822
60	0	40	0	0	20	41	0.029082	0.038844	0.048930	0.062678
60	0	40	0	0	20	42	0.050552	0.063442	0.076619	0.094040
60	5	25	10	15	5	56	0.093291	0.121227	0.149851	0.185926
60	5	25	10	15	5	57	0.168737	0.205859	0.242917	0.289904
80	0	60	0	0	20	61	0.020571	0.027262	0.034178	0.043733
80	0	60	0	0	20	62	0.035516	0.044177	0.052870	0.064465
80	20	50	0	0	10	71	0.107112	0.140298	0.172940	0.218301
80	20	50	0	0	10	72	0.186971	0.227849	0.268891	0.321012
80	35	40	0	0	5	76	0.248677	0.324994	0.398468	0.495168
80	35	40	0	0	5	77	0.454654	0.549467	0.646719	0.771039
80	10	30	20	15	5	76	0.104540	0.135988	0.166087	0.205324
80	10	30	20	15	5	77	0.189026	0.228394	0.266890	0.316156
100	0	80	0	0	20	81	0.016451	0.021594	0.027014	0.034692
100	0	80	0	0	20	82	0.028321	0.035183	0.041967	0.050858
100	30	60	0	0	10	91	0.108013	0.141537	0.173609	0.217909
100	30	60	0	0	10	92	0.188581	0.229242	0.269324	0.319907
100	40	55	0	0	5	96	0.204258	0.263513	0.322004	0.398843
100	40	55	0	0	5	97	0.370442	0.448971	0.525483	0.625227
100	30	30	10	20	10	91	0.111955	0.145320	0.178731	0.224922
100	30	30	10	20	10	92	0.195929	0.238947	0.279889	0.333703

Table A2
The percentiles data of $\widehat{U}_2$ distribution for $X_{(j)}$

n	r	k	l	m	s	j	0.90	0.95	0.975	0.99
10	0	3	0	0	7	4	1.125945	1.807184	2.789045	4.675009
10	0	3	0	0	7	5	1.926988	3.009241	4.505321	7.450012
10	0	8	0	0	2	9	0.341619	0.463249	0.596332	0.800317
10	0	8	0	0	2	10	0.690662	0.889023	1.104782	1.418024
10	2	5	0	0	3	8	0.854753	1.209239	1.630158	2.263632
10	2	5	0	0	3	9	1.607269	2.163889	2.823450	3.862621
10	3	4	0	0	3	8	0.857231	1.219519	1.624057	2.338495
10	3	4	0	0	3	9	1.602540	2.171974	2.823358	3.894914
10	1	3	1	2	3	8	0.535858	0.744663	0.979819	1.350916
10	1	3	1	2	3	9	0.994515	1.312484	1.671371	2.250541
10	1	2	2	1	4	7	0.626326	0.886672	1.189561	1.676252
10	1	2	2	1	4	8	1.119622	1.518678	1.973831	2.722859
13	0	10	0	0	3	11	0.224299	0.302451	0.387447	0.511807
13	0	10	0	0	3	12	0.418464	0.537054	0.657008	0.835174
13	0	8	0	0	5	9	0.228817	0.317286	0.416273	0.562407
13	0	8	0	0	5	10	0.404464	0.530687	0.666942	0.862566
13	3	6	0	0	4	10	0.678069	0.950597	1.254716	1.732797
13	3	6	0	0	4	11	1.239673	1.644747	2.096453	2.763499
13	4	7	0	0	2	12	0.925312	1.257150	1.627340	2.178310
13	4	7	0	0	2	13	1.925821	2.487037	3.082443	3.996974
13	2	5	1	1	4	10	0.476881	0.655567	0.862297	1.187076
13	2	5	1	1	4	11	0.864392	1.130554	1.425821	1.856429
13	3	6	1	1	2	12	0.683757	0.921912	1.184432	1.567907
13	3	6	1	1	2	13	1.423511	1.812982	2.209985	2.828038
20	0	10	0	0	10	11	0.138502	0.187599	0.244455	0.323612
20	0	10	0	0	10	12	0.238964	0.307889	0.382289	0.496495
20	0	16	0	0	4	17	0.125104	0.166941	0.210493	0.270785
20	0	16	0	0	4	18	0.230551	0.289223	0.348954	0.433756
20	2	4	2	2	10	11	0.289980	0.396636	0.513689	0.691903
20	2	4	2	2	10	12	0.496145	0.645765	0.803456	1.041065
40	0	30	0	0	10	31	0.046921	0.062344	0.078757	0.100786
40	0	30	0	0	10	32	0.082149	0.102351	0.123643	0.152041
40	10	20	0	0	10	31	0.187509	0.248863	0.310265	0.395684
40	10	20	0	0	10	32	0.326648	0.407463	0.485040	0.593071
40	8	15	3	10	4	37	0.207977	0.272013	0.335791	0.423044
40	8	15	3	10	4	38	0.384794	0.467108	0.550735	0.659227
60	0	50	0	0	10	51	0.031725	0.041907	0.052619	0.067134
60	0	50	0	0	10	52	0.055814	0.069054	0.082435	0.099854
60	0	40	0	0	20	41	0.026919	0.035758	0.044826	0.057369
60	0	40	0	0	20	42	0.046642	0.058305	0.070049	0.085775
60	5	25	10	15	5	56	0.089785	0.116649	0.143983	0.178583
60	5	25	10	15	5	57	0.162340	0.198148	0.233264	0.278437
80	0	60	0	0	20	61	0.019502	0.025802	0.032386	0.041330

(continued)

Table A2
(Continued)

n	r	k	l	m	s	j	0.90	0.95	0.975	0.99
80	0	60	0	0	20	62	0.033652	0.041705	0.049959	0.060742
80	20	50	0	0	10	71	0.111420	0.145947	0.179639	0.226672
80	20	50	0	0	10	72	0.194517	0.236813	0.278984	0.332760
80	35	40	0	0	5	76	0.340715	0.445412	0.545603	0.680659
80	35	40	0	0	5	77	0.623434	0.754996	0.886745	1.059126
80	10	30	20	15	5	76	0.097489	0.126730	0.154902	0.191224
80	10	30	20	15	5	77	0.176176	0.212840	0.248577	0.293974
100	0	80	0	0	20	81	0.015837	0.020760	0.025930	0.033227
100	0	80	0	0	20	82	0.027268	0.033765	0.040272	0.048652
100	30	60	0	0	10	91	0.119837	0.157001	0.192513	0.241898
100	30	60	0	0	10	92	0.209158	0.254306	0.298368	0.354761
100	40	55	0	0	5	96	0.268629	0.346453	0.422975	0.524522
100	40	55	0	0	5	97	0.487202	0.589738	0.690427	0.822849
100	30	30	10	20	10	91	0.120297	0.156290	0.191905	0.242041
100	30	30	10	20	10	92	0.210547	0.256663	0.301110	0.358294

Table A3
The percentiles data of $\widehat{U}_3$ distribution for $X_{(j)}$

n	r	k	l	m	s	j	0.90	0.95	0.975	0.99
10	0	3	0	0	7	4	1.608883	2.721678	4.335174	7.766941
10	0	3	0	0	7	5	2.821951	4.645655	7.286020	2.625970
10	0	8	0	0	2	9	0.415438	0.576626	0.760518	1.051593
10	0	8	0	0	2	10	0.856595	1.138579	1.447858	1.900054
10	2	5	0	0	3	8	1.287732	1.902335	2.658368	3.948120
10	2	5	0	0	3	9	2.484045	3.534942	4.807234	6.832533
10	3	4	0	0	3	8	1.292677	1.923213	2.676414	4.038137
10	3	4	0	0	3	9	2.481441	3.525552	4.791834	7.102913
10	1	3	1	2	3	8	0.788329	1.142375	1.562458	2.271983
10	1	3	1	2	3	9	1.503843	2.067356	2.781122	3.953569
10	1	2	2	1	4	7	1.036246	1.569153	2.280612	3.470068
10	1	2	2	1	4	8	1.934792	2.786440	3.943565	6.031424
13	0	10	0	0	3	11	0.268639	0.369248	0.483485	0.650099
13	0	10	0	0	3	12	0.507211	0.665167	0.834256	1.093373
13	0	8	0	0	5	9	0.294301	0.415125	0.559604	0.784012
13	0	8	0	0	5	10	0.529261	0.710224	0.913072	1.229491
13	3	6	0	0	4	10	1.010156	1.470474	2.018976	2.918500
13	3	6	0	0	4	11	1.893738	2.624943	3.507792	4.929825
13	4	7	0	0	2	12	1.219110	1.723033	2.312422	3.279273
13	4	7	0	0	2	13	2.596685	3.509463	4.578146	6.235811
13	2	5	1	1	4	10	0.783438	1.131463	1.547240	2.233809
13	2	5	1	1	4	11	1.455241	2.007740	2.697160	3.759295

(continued)

Table A3
(Continued)

n	r	k	l	m	s	j	0.90	0.95	0.975	0.99
13	3	6	1	1	2	12	1.024048	1.446484	1.923655	2.730271
13	3	6	1	1	2	13	2.188790	2.919691	3.781033	5.121392
20	0	10	0	0	10	11	0.178347	0.245756	0.323979	0.442596
20	0	10	0	0	10	12	0.309257	0.410631	0.520292	0.691798
20	0	16	0	0	4	17	0.142240	0.191015	0.244234	0.320397
20	0	16	0	0	4	18	0.263603	0.334755	0.408376	0.517137
20	2	4	2	2	10	11	0.485984	0.685521	0.925467	1.308522
20	2	4	2	2	10	12	0.851258	1.153520	1.504779	2.052176
40	0	30	0	0	10	31	0.051966	0.069362	0.088417	0.113779
40	0	30	0	0	10	32	0.091289	0.114727	0.139368	0.173413
40	10	20	0	0	10	31	0.251929	0.337695	0.427487	0.552761
40	10	20	0	0	10	32	0.442873	0.559259	0.678768	0.842530
40	8	15	3	10	4	37	0.257641	0.340726	0.422647	0.538615
40	8	15	3	10	4	38	0.479742	0.589852	0.702373	0.848675
60	0	50	0	0	10	51	0.033826	0.044828	0.056446	0.072470
60	0	50	0	0	10	52	0.059585	0.074052	0.088758	0.108073
60	0	40	0	0	20	41	0.029911	0.040093	0.050694	0.065241
60	0	40	0	0	20	42	0.052119	0.065548	0.079406	0.097732
60	5	25	10	15	5	56	0.106906	0.139594	0.173446	0.215759
60	5	25	10	15	5	57	0.194136	0.238175	0.282504	0.337432
80	0	60	0	0	20	61	0.021042	0.027963	0.035082	0.044998
80	0	60	0	0	20	62	0.036376	0.045310	0.054329	0.066468
80	20	50	0	0	10	71	0.132173	0.173655	0.214600	0.271911
80	20	50	0	0	10	72	0.230950	0.283157	0.335152	0.404494
80	35	40	0	0	5	76	0.337775	0.443119	0.549124	0.688864
80	35	40	0	0	5	77	0.620401	0.758094	0.898993	1.075803
80	10	30	20	15	5	76	0.127763	0.166956	0.205652	0.256236
80	10	30	20	15	5	77	0.231899	0.282722	0.332915	0.398145
100	0	80	0	0	20	81	0.016770	0.022076	0.027644	0.035607
100	0	80	0	0	20	82	0.028933	0.036024	0.043055	0.052245
100	30	60	0	0	10	91	0.137085	0.178875	0.221804	0.279193
100	30	60	0	0	10	92	0.238993	0.292765	0.345267	0.412394
100	40	55	0	0	5	96	0.273992	0.353812	0.433196	0.543029
100	40	55	0	0	5	97	0.497990	0.606280	0.711776	0.854817
100	30	30	10	20	10	91	0.149530	0.194642	0.241735	0.303730
100	30	30	10	20	10	92	0.262609	0.321812	0.379495	0.455206

Table A4
The percentiles data of $\widehat{U}_4$ distribution for $X_{(j)}$

n	r	k	l	m	s	j	0.90	0.95	0.975	0.99
10	0	3	0	0	7	4	1.151481	1.851704	2.865690	4.795962
10	0	3	0	0	7	5	1.970969	3.083778	4.600965	7.644822
10	0	8	0	0	2	9	0.379575	0.521806	0.681276	0.932792
10	0	8	0	0	2	10	0.776022	1.019825	1.287142	1.669262
10	2	5	0	0	3	8	1.069152	1.543437	2.122403	3.063773
10	2	5	0	0	3	9	2.031816	2.827447	3.740405	5.238370
10	3	4	0	0	3	8	1.067482	1.554482	2.124755	3.114511
10	3	4	0	0	3	9	2.026169	2.824255	3.751712	5.368628
10	1	3	1	2	3	8	0.616590	0.866977	1.160080	1.632357
10	1	3	1	2	3	9	1.156301	1.545349	1.997795	2.734704
10	1	2	2	1	4	7	0.689432	0.985376	1.330374	1.874105
10	1	2	2	1	4	8	1.244403	1.689974	2.210664	3.072054
13	0	10	0	0	3	11	0.244019	0.332777	0.432421	0.573764
13	0	10	0	0	3	12	0.458090	0.594576	0.739245	0.953147
13	0	8	0	0	5	9	0.244909	0.340996	0.451168	0.616949
13	0	8	0	0	5	10	0.435766	0.575554	0.727715	0.956080
13	3	6	0	0	4	10	0.882208	1.260661	1.715230	2.425762
13	3	6	0	0	4	11	1.636582	2.243070	2.941178	4.039124
13	4	7	0	0	2	12	1.435964	2.086102	2.831623	4.150479
13	4	7	0	0	2	13	3.087689	4.285415	5.733665	8.080045
13	2	5	1	1	4	10	0.575818	0.805406	1.072823	1.498267
13	2	5	1	1	4	11	1.055841	1.408208	1.816994	2.413378
13	3	6	1	1	2	12	0.952730	1.339496	1.778243	2.468213
13	3	6	1	1	2	13	2.032407	2.687218	3.456675	4.673351
20	0	10	0	0	10	11	0.144297	0.195873	0.255863	0.340918
20	0	10	0	0	10	12	0.248815	0.322439	0.401536	0.525099
20	0	16	0	0	4	17	0.132960	0.178270	0.226451	0.295959
20	0	16	0	0	4	18	0.246340	0.310967	0.377813	0.474480
20	2	4	2	2	10	11	0.319509	0.437563	0.569956	0.770382
20	2	4	2	2	10	12	0.547889	0.717535	0.896565	1.172809
40	0	30	0	0	10	31	0.048600	0.064684	0.082079	0.105274
40	0	30	0	0	10	32	0.085220	0.106602	0.129155	0.159517
40	10	20	0	0	10	31	0.282031	0.379259	0.486156	0.638072
40	10	20	0	0	10	32	0.499665	0.634700	0.777238	0.980392
40	8	15	3	10	4	37	0.291018	0.385185	0.480928	0.617301
40	8	15	3	10	4	38	0.544320	0.673119	0.805277	0.983333
60	0	50	0	0	10	51	0.032653	0.043213	0.054354	0.069595
60	0	50	0	0	10	52	0.057499	0.071340	0.085348	0.103809
60	0	40	0	0	20	41	0.027524	0.036709	0.046123	0.059063
60	0	40	0	0	20	42	0.047802	0.059890	0.072177	0.088481
60	5	25	10	15	5	56	0.102322	0.133549	0.165190	0.206084
60	5	25	10	15	5	57	0.185512	0.227489	0.269386	0.321875
80	0	60	0	0	20	61	0.019947	0.026421	0.033131	0.042377

(continued)

Table A4
(Continued)

n	r	k	l	m	s	j	0.90	0.95	0.975	0.99
80	0	60	0	0	20	62	0.034451	0.042765	0.051206	0.062448
80	20	50	0	0	10	71	0.183705	0.243337	0.305340	0.393372
80	20	50	0	0	10	72	0.323793	0.402328	0.485835	0.593650
80	35	40	0	0	5	76	1.022273	1.439541	1.955282	2.792232
80	35	40	0	0	5	77	1.963860	2.649698	3.459104	4.874712
80	10	30	20	15	5	76	0.119494	0.155638	0.191754	0.237721
80	10	30	20	15	5	77	0.216451	0.262737	0.309566	0.366836
100	0	80	0	0	20	81	0.016155	0.021214	0.026529	0.034072
100	0	80	0	0	20	82	0.027828	0.034538	0.041214	0.049966
100	30	60	0	0	10	91	0.236382	0.316016	0.400905	0.520014
100	30	60	0	0	10	92	0.420089	0.529323	0.639775	0.800463
100	40	55	0	0	5	96	0.749712	1.025821	1.350001	1.864737
100	40	55	0	0	5	97	1.422527	1.859327	2.369456	3.186211
100	30	30	10	20	10	91	0.237454	0.316421	0.400959	0.520610
100	30	30	10	20	10	92	0.421915	0.531830	0.641117	0.809200

Table A5
The percentiles data of $\widehat{U}_a$ distribution for $X_{(j)}$

n	r	k	l	m	s	j	0.90	0.95	0.975	0.99
10	0	3	0	0	7	4	2.429833	3.884042	6.000749	9.953094
10	0	3	0	0	7	5	4.138215	6.444420	9.604718	15.893063
10	0	8	0	0	2	9	0.878576	1.169411	1.481491	1.929139
10	0	8	0	0	2	10	1.732412	2.181317	2.656846	3.327461
10	2	5	0	0	3	8	1.008783	1.420539	1.888689	2.628249
10	2	5	0	0	3	9	1.885141	2.518690	3.262318	4.456112
10	3	4	0	0	3	8	1.013620	1.423626	1.897605	2.715873
10	3	4	0	0	3	9	1.875346	2.521363	3.264253	4.525454
10	1	3	1	2	3	8	0.857669	1.191790	1.578126	2.220072
10	1	3	1	2	3	9	1.591620	2.115796	2.702158	3.681147
10	1	2	2	1	4	7	1.169301	1.808686	2.730098	4.595618
10	1	2	2	1	4	8	2.153508	3.249308	4.758866	8.031949
13	0	10	0	0	3	11	0.616219	0.818953	1.032451	1.332502
13	0	10	0	0	3	12	1.134672	1.423598	1.710585	2.110300
13	0	8	0	0	5	9	0.617341	0.839394	1.081052	1.426567
13	0	8	0	0	5	10	1.072864	1.385328	1.717790	2.173359
13	3	6	0	0	4	10	0.704400	0.981970	1.293833	1.761592
13	3	6	0	0	4	11	1.278436	1.695298	2.145875	2.837239
13	4	7	0	0	2	12	0.806103	1.085826	1.390278	1.835851
13	4	7	0	0	2	13	1.665137	2.113019	2.611115	3.352772
13	2	5	1	1	4	10	0.595447	0.815059	1.068037	1.449737

(continued)

Table A5
(Continued)

n	r	k	l	m	s	j	0.90	0.95	0.975	0.99
13	2	5	1	1	4	11	1.075171	1.399081	1.756683	2.299445
13	3	6	1	1	2	12	0.697125	0.929402	1.179045	1.537952
13	3	6	1	1	2	13	1.430224	1.800205	2.180568	2.741729
20	0	10	0	0	10	11	0.403657	0.542627	0.690618	0.903539
20	0	10	0	0	10	12	0.686688	0.871494	1.061383	1.345981
20	0	16	0	0	4	17	0.391437	0.514785	0.636402	0.800235
20	0	16	0	0	4	18	0.710061	0.870636	1.031083	1.239741
20	2	4	2	2	10	11	0.457084	0.643474	0.862236	1.221303
20	2	4	2	2	10	12	0.792695	1.060140	1.379064	1.892893
40	0	30	0	0	10	31	0.175387	0.229385	0.282974	0.356898
40	0	30	0	0	10	32	0.301198	0.370072	0.438520	0.525298
40	10	20	0	0	10	31	0.182691	0.241560	0.301287	0.383467
40	10	20	0	0	10	32	0.317068	0.393065	0.466655	0.568042
40	8	15	3	10	4	37	0.242214	0.315173	0.386977	0.484742
40	8	15	3	10	4	38	0.446239	0.541538	0.634735	0.753612
60	0	50	0	0	10	51	0.131917	0.172110	0.212294	0.262150
60	0	50	0	0	10	52	0.228395	0.277737	0.324524	0.386409
60	0	40	0	0	20	41	0.108635	0.142665	0.175657	0.220025
60	0	40	0	0	20	42	0.186019	0.228381	0.270367	0.324718
60	5	25	10	15	5	56	0.163435	0.210540	0.260313	0.321630
60	5	25	10	15	5	57	0.294847	0.357957	0.420505	0.495252
80	0	60	0	0	20	61	0.085832	0.112066	0.137474	0.172343
80	0	60	0	0	20	62	0.145818	0.178012	0.208948	0.250070
80	20	50	0	0	10	71	0.114208	0.148920	0.184303	0.230278
80	20	50	0	0	10	72	0.199059	0.242317	0.283440	0.337633
80	35	40	0	0	5	76	0.170275	0.221427	0.273045	0.338932
80	35	40	0	0	5	77	0.309794	0.377808	0.441526	0.525204
80	10	30	20	15	5	76	0.132700	0.171705	0.209124	0.258594
80	10	30	20	15	5	77	0.239123	0.288233	0.335466	0.395676
100	0	80	0	0	20	81	0.073274	0.095275	0.117025	0.146032
100	0	80	0	0	20	82	0.124923	0.151890	0.177643	0.211562
100	30	60	0	0	10	91	0.101833	0.133259	0.162852	0.205107
100	30	60	0	0	10	92	0.177375	0.216268	0.252996	0.300836
100	40	55	0	0	5	96	0.155875	0.201068	0.245817	0.304968
100	40	55	0	0	5	97	0.282746	0.342202	0.400831	0.475835
100	30	30	10	20	10	91	0.090444	0.117285	0.144743	0.182476
100	30	30	10	20	10	92	0.157932	0.193016	0.225770	0.267999

Appendix 3

A Simple Algorithm in SPARKS Language

```
// 1. From the simple algorithm, it is easily
for generating the results of Tables 1–4 and
Tables A1–A5. //
// 2. After setting loopnum=600,000, the
Tables A1–A4 will be generated.//
// 3. After creating the Tables A1–A5,
modifing a little bit, and setting
loopnum=10,000//
//    the results of simulation in
Table 2 will be obtained.//
// 4. Substituting X(i) in the procedure
randvariableX with the X(i) of illustrated
examples //
//    and set loopnum=1, the results of
Tables 3–4 will be easily obtained.//
// Set n, r, k, l, m, s, j   values.//
global  n, r, k, l, m, s, j, sume, tempe,
X(1 : n), P(1 : n), Q(1 : n), E(1 :
n), ALPHA(1 : n), BETA(1 : n);
global w1, w2, w3, w4, wa;
local integer i, loopnum, u1j1, u1j2, u2j1,
u2j2, u3j1, u3j2;
local integer u4j1, u4j2, uaj1, uaj2;
  call expectedvalueE
  for i ← 1 to loopnum do
    call randvariableX
    call W1
    u1j1 ← (X(n − s + 1) − X(n − s))/w1
    u1j2 ← (X(n − s + 2) − X(n − s))/w1
    print (file, u1j1, u1j2)
    call W2
    u2j1 ← (X(n − s + 1) − X(n − s))/w2
    u2j2 ← (X(n − s + 2) − X(n − s))/w2
    print (file, u2j1, u2j2)
    call W3
    u3j1 ← (X(n − s + 1) − X(n − s))/w3
    u1j2 ← (X(n − s + 2) − X(n − s))/w3
    print (file, u3j1, u3j2)
    call W4
    u4j1 ← (X(n − s + 1) − X(n − s))/w4
    u4j2 ← (X(n − s + 2) − X(n − s))/w4
    print (file, u4j1, u4j2)
    call Wa
    uaj1 ← (X(n − s + 1) − X(n − s))/wa
    uaj2 ← (X(n − s + 2) − X(n − s))/wa
    print (file, uaj1, uaj2)
  repeat
// After generating 600,000 data, sort those
data and get the percentiles of Û_k
(k = 1...4, a), respectively. //
end
procedure expectedvalueE
  // generate the expected values of X(i) //
  integer i;
  for i ← 1 to n do
     E(i) = f( ∫_{−∞}^{∞} x_(i) f(x_(i)) dx_(i) )
     // f(x_(i)) is the pdf of the ith order
statistics.//
  repeat
   E(1) = 0
   sume = 0
  for i ← 1 to n do
     if i ≠ r + 1 then sume ← sume + E(i)
     endif
  repeat
end expectedvalueE
procedure randvariableX
  // generate the random variables of X(i)
and then sort them//
  integer i, j, temp;
  for i ← 1 to n do
     fx ← RND    // RND is a function for
geanerating random variables.//
     X(i) ← LOG(−LOG(1 − fx))    // LOG
is a function for geanerating logarithmic
values.//
  repeat
  for i ← 1 to n − 1 do
     for j ← i + 1 to n do
       if X(i) > X(j) then
        temp ← X(i); X(i) ← X(j)
        X(j) ← temp
       endif
     repeat
  repeat
end randvariableX
procedure W1
  // generate the pivotal quantities of
Ŵ_1 //
  integer i, b;
  w1 ← 0; tempe ← 0; b ← r + k + l
  for i ← 2 to k do
     tempe ← tempe + E(r + i)
  repeat
  for i ← 2 to m do
     tempe ← tempe + E(b + i)
  repeat
```

for $i \leftarrow 2$ **to** k **do**
$w1 \leftarrow w1 + (E(r+i)/tempe) * (X(r+i) - X(r+1))$
repeat
for $i \leftarrow 2$ **to** m **do**
$w1 \leftarrow w1 + (E(b+i)/tempe) * (X(b+i) - X(r+1))$
repeat
end $W1$
procedure $W2$
// generate the pivotal quantities of $\widehat{W}_2$ //
integer i, b;
$w2 \leftarrow 0$
for $i \leftarrow 1$ **to** r **do**
$w2 \leftarrow w2 + (E(i)/sume) * (X(r+2) - X(r+1))$
repeat
for $i \leftarrow 2$ **to** k **do**
$w2 \leftarrow w2 + (E(r+i)/sume) * (X(r+i) - X(r+1))$
repeat
$b \leftarrow r+k$
for $i \leftarrow 1$ **to** l **do**
$w2 \leftarrow w2 + (E(b+i)/sume) * ((X(b+l+1) - X(r+1)) + (X(b) - X(r+1)))/2$
repeat
$b \leftarrow r+k+l$
for $i \leftarrow 2$ **to** m **do**
$w2 \leftarrow w2 + (E(b+i)/sume) * (X(b+i) - X(r+1))$
repeat
$b \leftarrow r+k+l+m$
for $i \leftarrow 2$ **to** s **do**
$w2 \leftarrow w2 + (E(b+i)/sume) * (X(b) - X(r+1))$
repeat **end** $W2$
procedure $W3$
// generate the pivotal quantities of $\widehat{W}_3$ //
integer i, b;
$w3 \leftarrow 1$
for $i \leftarrow 2$ **to** k **do**
$w3 \leftarrow w3 * POW((X(r+i) - X(r+1)), (E(r+i)/sume))$ // POW is a function of power //
repeat
$b \leftarrow r+k+l$
for $i \leftarrow 1$ **to** m **do**
$w3 \leftarrow w3 * POW((X(b+i) - X(r+1)), (E(b+i)/sume))$
repeat
end $W3$
procedure $W4$
// generate the pivotal quantities of $\widehat{W}_4$ //
integer i, b;
$w4 \leftarrow 1$
for $i \leftarrow 1$ **to** r **do**
$w4 \leftarrow w4 * POW((X(r+2) - X(r+1)), (E(i)/sume))$
repeat
for $i \leftarrow 2$ **to** k **do**
$w4 \leftarrow w4 * POW((X(r+i) - X(r+1)), (E(r+i)/sume))$
repeat
$b \leftarrow r+k$
for $i \leftarrow 2$ **to** l **do**

$w4 \leftarrow w4 * POW((((X(b+l+i) - X(r+1)) + (X(b) - X(r+1)))/2), (E(b+i)/sume))$
repeat
$b \leftarrow r+k+l$
for $i \leftarrow 1$ **to** m **do**
$w4 \leftarrow w4 * POW((X(b+i) - X(r+1)), (E(b+i)/sume))$
repeat
$b \leftarrow r+k+l+m$
for $i \leftarrow 1$ **to** s **do**
$w4 \leftarrow w4 * POW((X(b) - X(r+1)), (E(b+i)/sume))$
repeat
end $W4$
procedure Wa
// According to Eq. (9), generate the pivotal quantities of $\widehat{W}_a$.//
end Wa

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