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Multifractal analyses of music sequences

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Abstract

Multifractal analysis is applied to study the fractal property of music. In this paper, a method is proposed to transform both the melody and rhythm of a music piece into individual sets of distributed points along a one-dimensional line. The structure of the musical composition is thus manifested and characterized by the local clustering pattern of these sequences of points. Specifically, the local Hölder exponent and the multifractal spectrum are calculated for the transformed music sequences according to the multifractal formalism. The observed fluctuations of the Hölder exponent along the music sequences confirm the non-uniformity feature in the structures of melodic and rhythmic motions of music. Our present result suggests that the shape and opening width of the multifractal spectrum plot can be used to distinguish different styles of music. In addition, a characteristic curve is constructed by mapping the point sequences converted from the melody and rhythm of a musical work into a two-dimensional graph. Each different pieces of music has its own unique characteristic curve. This characteristic curve, which also exhibits a fractal trait, unveils the intrinsic structure of music.

Keywords: Music; Fractal; Multifractal analysis; Multifractal spectrum; Hölder exponent

1. Introduction

Nature is full of irregular patterns and complicated phenomena. Despite their complicated appearances, ‘self similarity’, that is, the similarity between the whole and a small portion of a system, can be observed in many configurations and phenomena upon closer investigation. Geometry with such scale-invariant features has now been categorized and designated as ‘fractal’ in literature [1]. Many geometries existing in nature are fractal, e.g., a mountain’s profile and the shape of snowflakes. Music, whose origin may be attributed to imitating the harmony of nature’s sound, also demonstrates a fractal property like many other naturally occurring fluctuations do.

Music can be used to express human feelings and emotions toward nature. A few musical notes can be aligned by a composer’s will into a beautiful and pleasant song; whereas the same notes can be arranged into an annoying or discordant noise if randomly aligned. So what is the mystique of music? This is an issue that has been investigated for hundreds of years, but has not been concluded so far. Fractal theory [1], developed in the 1970s, provides an innovative tool for the analysis of a sequence of symbols. By applying fractal tools in the study of music, researchers, including Voss and Hsu, were surprised to discover that the self-similarity property, which is ubiquitous in nature, also exists in music. Such an observation may be regarded as the first step toward a further understanding of what music is and explaining how music simulates the harmony of nature.

1.1. Frequency ratio between music tones

When comparing two tones, a frequency ratio of small number integers (e.g. 1:2 (an octave), 2:3 (a fifth), etc., under the circumstance of ‘just intonation’) indicates a more harmonious sound than a ratio of larger number integers (e.g. 5:6 (a minor third), 15:16 (a minor second), etc.). Just intonation is a system of tuning in which all of the intervals can be represented by ratios of whole numbers, with a strongly-implied preference for the smallest numbers compatible with a given musical purpose. Unfortunately this definition, while

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accurate, does not convey much to those who are not already familiar with the art and science of tuning. The piano and almost all modern keyboard instruments follow the twelve-tone scale; i.e. an octave with a frequency ratio of 1:2 is divided geometrically by even intervals into 12 semitones, each corresponding to one of the seven white or five black keys on the piano, and their frequency and fundamental frequency f_0 satisfy the exponential function of $f_j/f_0 = 2^{j/12}$. The twelve-tone scale differs from just intonation in frequency ratio; e.g. a perfect fourth consists of 5 semitones with a frequency ratio of $2^{5/12} = 1.3348$, which is close to 4/3; a perfect fifth consists of 7 semitones with a frequency ratio of $2^{7/12} = 1.4983$, which is nearly 3/2. Both are ratios of smaller integers. A diminished fifth, however, has 6 semitones with a frequency ratio of $2^{6/12} = 1.4142$, which is almost 1000/707. This is not a ratio of small integers. Therefore, such an interval has been traditionally considered dissonant and is rarely used in classical pieces.

1.2. Music as $1/f$ noise

Before discussing the relationship between music and fractal theory, let us focus on a particular type of noise — $1/f$ noise first. Mandelbrot proposed that there is a kind of sound in which the quality is unaffected by changes in play speed, and called this sound 'scaling noise' [1]. The plainest example of scaling noise is 'white noise'. Suppose a time series is produced in accordance with temporal variations of white noise, a calculation of its power spectral density $S(f)$ reveals that the relationship between $S(f)$ and f can be stated as $S(f) \propto f^{-\beta}$, where scaling exponent $\beta = 0$, indicating its monotonousness at whatever play speed. In other words, white noise is a mixture of frequency components from a wide range that are randomly and completely combined; its features are utmost randomness and totally unrelated points. Brownian noise is another type of scaling noise with scaling exponent $\beta = 2$. It depicts Brownian movement or random walk, with the strongest correlation among points within a characteristic time scale.

On the other hand, after conducting a spectral analysis on various types of music, including classical music (Bach, Mozart, Beethoven ...) and modern jazz, Voss and Clarke [2,3] discovered that musical works of various melodies and styles share a similar tendency toward a $1/f$ spectrum. In fact, music featuring a $1/f$ spectrum happens to be a $1/f$ noise intermediary between the flat spectrum of white noise and the steep $1/f^2$ spectrum of Brownian noise. It is a kind of scaling noise, too. However, neither white noise nor Brownian noise can be called music; the former is so random and unassociated that it becomes uninteresting, while the latter has over-emphasized connections and lacks charm. Only $1/f$ noise can merge the randomness and orderliness into a naturally pleasant and attractive whole [4,5].

1.3. Fractal geometry in music

Observation of time series of $1/f$ noise with various time scales reveals statistical self-similarity. That is to say, any enlargement or reduction of the timeline would not affect the tendency of fluctuation. Mandelbrot called such behavior scale invariance. Furthermore, $1/f$ noise features a long-range correlation, or retaining memory over a rather long period of time. Coincidently, nature is saturated with the $1/f$ phenomenon, as seen in a mountain contour and the fluctuation of a river's water level, whose variations also have the traits of scale invariance and long-range correlation. The spectral analysis in the study by Voss and Clarke substantiated the assumption that music imitates characteristics of temporal variations demonstrated by nature and the universe, and that music features fractal geometry.

As mentioned above, Voss and Clarke, from their analysis on the power spectrum $S(f)$ of musical signals of various styles, observed fractal distribution approximating to $1/f$ in power spectra of both loudness and frequency fluctuation (waves of melody). However, they also pointed out that such a phenomenon is not found in all ranges of frequency; instead, it is only so between 100 Hz and 10 kHz. In cases of high frequency (100 Hz–2 kHz), $S(f)$ is not molded as $1/f$. Hence Voss and Clarke suggested that, within a certain range, signal fluctuations of most musical works feature long range correlation, and the exponents of the power spectrum may also be associated with fractal content of music.

In the 1990s, Hsu and Hsu [6] discovered from analysis of music scores by Bach and Mozart that, in general, the difference in pitch j between two successive notes (i.e. the melody) and the frequency of their appearance F have an exponential relation, which can be stated as $F \propto j^{-D}$, where D is dimension. Values of the exponent D in various musical scores range between 1 and 3, but they are not integers. As the dimension is not a whole number, the frequency of pitch variation in music can be categorized as fractal geometry. In order to visualize music, Hsu and Hsu [7] used the j value to represent each musical note in a score, marked them in order of appearance on coordinate axes (x, y), forming a curve, and then diminished the sequence length by labeling points at intervals of 2, 4, and 8 ... points. The reduced curve looked much the same as the original one, and the style remained unaffected. Therefore, musical scores share the feature of self-similarity with fractal geometry [8].

In addition, in a recent study Shi [9] employed the calculation method of the Hurst exponent to examine the pitch sequence fashioned in folk songs and piano pieces. Their results indicated that music sequences have the property of long range correlation and the fundamental principle of music is the balance between repetition and contrast. Further, Bigerelle and Iost [10] applied the 'Variance Method' to study the fractal dimensions in 180 musical works of various styles. Based on statistical results, they proposed that various music pieces could be categorized by fractal dimension. Madison [11] used a similar approach to study different musical scores with Hurst exponents, which were found thereafter to play an important role in the emotional expression of musical performance. The study by Manaris et al. [12] of a 220-piece corpus (baroque, classical, romantic, 12-tone, jazz, rock, DNA strings, and random music) revealed that esthetically pleasing music might be describable under the Zipf–Mandelbrot law. Gunduz and Gunduz [13] studied the mathematical structures of six songs by treating them as complex systems. They also calculated the fractal dimension of a scattering diagram constructed from the six songs' melody.

From the above literature review, it is noticed that the Fourier power spectrum, as well as the analysis methods used by many previous investigators to compute the Hurst exponent and the value of fractal dimension only refer to the 'mean' properties of the overall sequence. However, it is well-established experience that naturally evolving geometries and phenomena are rarely characterized by a single scaling ratio; different parts of a system may be scaling differently. That is, the clustering pattern is not uniform over the whole system. Such a system is better characterized as 'multifractal' [1]. A multifractal can be loosely thought of as an interwoven set constructed from sub-sets with different local fractal dimensions. Real world systems are mostly multifractal in nature. Music too, as will be shown later in this paper, has non-uniform property in its movement. It is therefore necessary to re-investigate the musical structure from the viewpoint of the multifractal theory.

2. Multifractal analysis

2.1. Multifractal formalism

There are two common approaches for multifractal formalism:

(i) Generalized dimensions [14]

Suppose points with a total number of N are distributed in the space. Weighing local mass density $p_i(r)$ of points with different exponents q (moment) would lead to the definition of generalized dimensions:

$$D_q = \frac{1}{q-1} \lim_{r\to 0} \frac{\log \sum_i p_i(r)^q}{\log r} \tag{1}$$

where $p_i(r) = N_i(r)/N$ is the portion of points that fall within the ith sub-cover with size r, q is the given weight, and D_q is the generalized dimension. If local densities of point distributions in a fractal set are scattered unevenly, its D_q value varies with the given weight q. When $q < 1$, D_q reflects the fractal dimension of low-density point distributions in the set (or dispersive areas); while for $q > 1$, D_q reflects the fractal dimension of high-density point distributions in the set (or dense areas). By definition, D_0 is just the conventional box-counting dimension, D_1 is the information dimension and D_2 is the correlation dimension.

(ii) Multifractal spectrum [15,16]

The other multifractal formalism is to calculate the local scaling exponent of point distribution, also called the Hölder exponent α:

$$\alpha = \lim_{r\to 0} \frac{\log p_i(r)}{\log r}. \tag{2}$$

The physical significance of α is that $\alpha = 1$ indicates uniform distribution of points, while $\alpha < 1$ and $\alpha > 1$ represent 'dense inside and dispersive outside' and 'dispersive inside and dense outside' types of point distribution, respectively. Now let $n(\alpha)\mathrm{d}\alpha$ denote the number of sub-covers with the local scaling exponent ranging between α and $\alpha + \mathrm{d}\alpha$. If the original point set features a multifractal distribution, then $n(\alpha)$ and the size of sub-cover r again has a power-law relation:

$$n(\alpha) \sim r^{-f(\alpha)}. \tag{3}$$

In this equation, the power $f(\alpha)$ can be viewed as the fractal dimension of the set formed by sub-sets with a local scaling exponent of α. The correlation diagram of $f(\alpha)$ and α is called the multifractal spectrum of the point distribution.

By an analogy to well-known relationships in thermodynamics [15,16], it is induced that $f(\alpha)$ and α are related to the generalized dimension D_q and q via a Legendre transformation:

$$\alpha = \frac{\mathrm{d}}{\mathrm{d}q}[(q-1)D_q] \tag{4}$$

$$f(q) = q\frac{\mathrm{d}}{\mathrm{d}q}[(q-1)D_q] - (q-1)D_q. \tag{5}$$

Common approaches first calculate the generalized dimension D_q and then use Eqs. (4) and (5) to find α and $f(\alpha)$. The prerequisite of this procedure, however, is that D_q must be a smooth function of q. For signals adopted from nature, such postulation is not appropriate. Hence other researchers [17,18] proposed another method to obtain the multifractal spectrum $f(\alpha)$ directly from the weighted $p_i(r)$; i.e. set

$$\mu_i(r,q) = p_i(r)^q \Big/ \sum_i p_i(r)^q \tag{6}$$

then

$$\alpha(q) = \lim_{r\to 0} \frac{\sum_i \mu_i(r,q) \log p_i(r)}{\log r} \tag{7}$$

$$f(q) = \lim_{r\to 0} \frac{\sum_i \mu_i(r,q) \log \mu_i(r,q)}{\log r}. \tag{8}$$

In this paper, the direct formulations (7) and (8) are used to determine the multifractal spectrum of point distribution in music.

2.2. Conversion of musical melody and rhythm into sequences of points

Melody and rhythm are the two important elements of music. Conventionally, melody is defined as successive changes in pitch (tone) in an ordered arrangement of sounds, and rhythm is defined as successive changes in tone duration of the arranged sounds [6–8]. Before applying the multifractal analysis, the melody and rhythm of a music piece must be converted into sequences that are amenable to the multifractal formalism. The method we propose in this study is stated as follows.

Point distribution of musical melody is constructed by dividing an octave evenly into 12 pitches, as per defined in the twelve-tone scale. The first note of the music piece is chosen as the base point and a black point is placed at the first position of an imaginary line. If the absolute value of pitch difference between the second note and the first note (i.e. melody) is m,

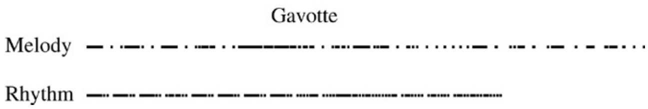

Fig. 1. Point distributions converted from the melody and rhythm of the music score *Gavotte*.

another black point is placed at the $(m + 1)$th position of the line. This procedure is then repeated until the point distribution diagram of musical melody is completed.

As for the construction of point sequence for rhythm, a shortest measure of time is first selected, e.g., the sixteenth note, and a black point is marked on the first position of the line. If the first note of the music piece is a sixteenth note, a black point is placed at the 2nd position of the line. If the second note is an eighth note, another black point is placed at the 4th position of the line, indicating that the temporal interval between this note and the following note (i.e. rhythm) is $4-2 = 2$ beats. Continuous repetition of this process completes the point distribution diagram of rhythm. As an example, Fig. 1 shows the one-dimensional point distributions of melody and rhythm transformed from the music score *Gavotte* by Gossec. Just by observation, we can easily discern the non-uniform structures in both the melodic and rhythmic motions of the music piece.

2.3. Calculations of α and $f(\alpha)$

Calculation of the Hölder exponent α is performed as follows. Choosing any position i (whether a black point or blank) on the line as the center of an interval, after selecting various interval sizes r, the portion of black points of total number N that reside in the interval of radius r, denoted by $p_i(r)$, is calculated. The results are plotted on the diagram of $\log r - \log p_i(r)$, and the slope of the curve represents the local Hölder exponent at this center position. Similarly, the procedure can be repeated at every position i on the line, and the variation of the Hölder exponent along the line is obtained. The variation curve of the Hölder exponent against position reveals how the local point distribution changes in density.

The multifractal spectrum of the point distribution is determined according to Eqs. (7) and (8). First, a point distribution resembling the one shown in Fig. 1 is covered by boxes with size r. If the probability of a black point falling into the ith box is $p_i(r)$, then μ_i can be derived from Eq. (6) with specified weighting exponent q, and values of $\sum \mu_i \log p_i$ and $\sum \mu_i \log \mu_i$ can be calculated based on the given box size r and weighting q. Next, the value of r is changed, and the corresponding values of $\sum \mu_i \log p_i$ and $\sum \mu_i \log \mu_i$ are re-calculated. Continuing this way, the results are plotted on diagrams of $\log r - \sum \mu_i \log p_i$ and $\log r - \sum \mu_i \log \mu_i$. Proper scaling regions are identified in these diagrams, and the slopes of the curves within the scaling ranges are calculated by the least-square fitting method. These slopes are the values of $\alpha(q)$ and $f(q)$, respectively. The whole process is then repeated for various values of weighting exponent q chosen between $-\infty$ and $+\infty$. The curve traced by $\alpha - f(\alpha)$ is the multifractal spectrum of the point distribution.

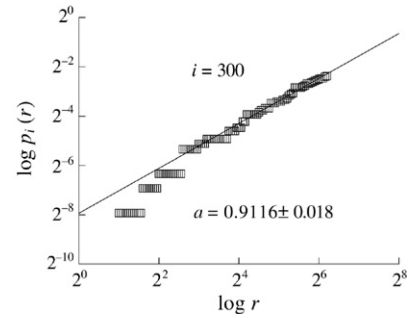

Fig. 2. Scaling of $p_i(r)$ with box size r.

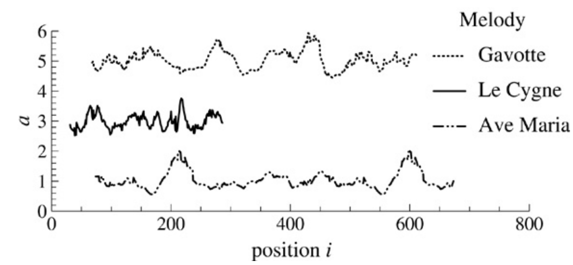

Fig. 3. Variation of Hölder exponents along the melody sequences converted from three different music scores. Data values of *Le Cygne* and *Gavotte* have been shifted upward by 2 and 4 units respectively for clarity.

3. Results and discussion

Three music scores were analyzed in this study: (I) Gossec's *Gavotte*, (II) Saint-Saëns' *Le Cygne*, and (III) *Ave Maria* by Bach and Gounod. After the melody and rhythm of the scores were converted into point distributions, the scaling exponent of local point density (i.e. the Hölder exponent) and the multifractal spectrum of each sequences of points were calculated by using the methods described in the previous section.

Fig. 2 shows a typical log–log plot of point density $p_i(r)$ vs. radius r of the interval centered at position $i = 300$ of the melody sequence transformed from the score *Ave Maria*. The smallest radius of the interval has a width of 2, while the largest radius can extend to a length of dozens of positions. To avoid the boundary effect (since the sequence is finite in length), the largest radius of the box is limited to 1/10 of the total length of the sequence, in this case, about 74. The local Hölder exponent α is then obtained from a linear fitting of the data points within the scaling range.

Detailed temporal organizations of the melodic and rhythmic motions of music can be analyzed by inspecting the local values of the Hölder exponent α. Figs. 3 and 4 show the variations of the Hölder exponent α along the sequences converted from the melodies and rhythms of the three different musical scores mentioned above. Irregular fluctuations of the curves around the value of $\alpha = 1$ are apparent in these figures. The Hölder exponent (also called the local crowding index) defined in Eq. (2) reflects the invariant scaling nature of the population density of point distribution in a small region centered at position i with those in the vicinity of increasing sizes. Variation in α value with position i signifies changes in the local clustering pattern

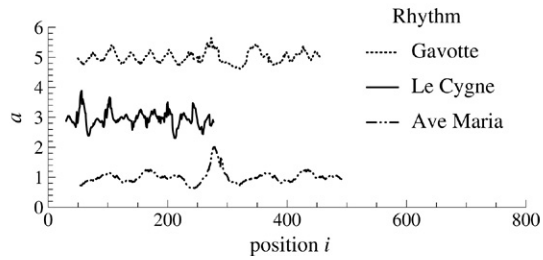

Fig. 4. Variation of Hölder exponents along the rhythm sequences converted from three different music scores. Data values of *Le Cygne* and *Gavotte* have been shifted upward by 2 and 4 units respectively for clarity.

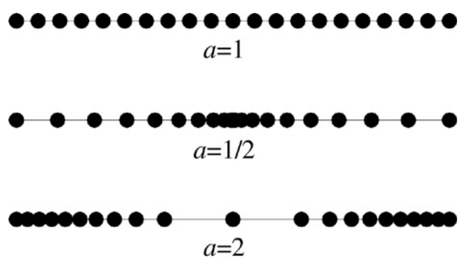

Fig. 5. Typical point distributions with different α values.

of point distribution along the line. The geometric interpretation of the Hölder exponent α is most instructively illustrated by Fig. 5 where point distributions corresponding to the special cases $\alpha = 1$, $\alpha = 1/2$ and $\alpha = 2$ are compared. It is seen that an α value less than one denotes a densely occupied region surrounded by a sparse vicinity, while an α value greater than one represents a less-populated region surrounded by a dense vicinity. The fluctuating α curves shown in Figs. 3 and 4 clearly suggest that arrangements of the melody and rhythm of music are highly non-uniform in structure. In obtaining the α curve, once again, only the middle 1/10 to 9/10 portion of the sequence was analyzed, and the maximal box size r was limited to 1/10 of the total length of the sequence so as to avoid edge effect.

The characteristics and cragginess of the α curve can be further quantitatively analyzed by inspecting the Hurst exponent of the sequence. Mandelbrot and Van Ness [19] generalized the expression of the diffusion law of a Brownian motion $x(t)$ (a random-walk sequence) into the form

$$\overline{\Delta x_H(T)} = \langle |x_H(t+T) - x_H(t)|^2 \rangle^{1/2} \propto T^H \tag{9}$$

where $\overline{\Delta x_H(T)}$ denotes the mean distance traveled in the time span T and H is the Hurst exponent, which ranges between 0 and 1. The corresponding motion (time sequence) $x_H(t)$ is now favorably called the 'fractional Brownian motion' (fBm). For $H > 1/2$, the graph of $x_H(t)$ is less rugged-looking (smoother) than that of the Brownian motion ($H = 1/2$), and $x_H(t)$ tends to be increasing in the future if it is increasing in the past (i.e. persistence in trend). For $H < 1/2$, the graph of $x_H(t)$ is more rugged-looking than that of $H = 1/2$, and $x_H(t)$ tends to be decreasing in the future if it is increasing in the past (i.e. anti-persistence in trend). The Hurst exponent H was calculated for the above various α curves, and the results are summarized in

Table 1
The Hurst exponent H for α curves obtained from various music pieces

	Gavotte	*Le Cygne*	*Ave Maria*
Melody	0.5748	0.4221	0.6377
Rhythm	0.5600	0.4344	0.7361

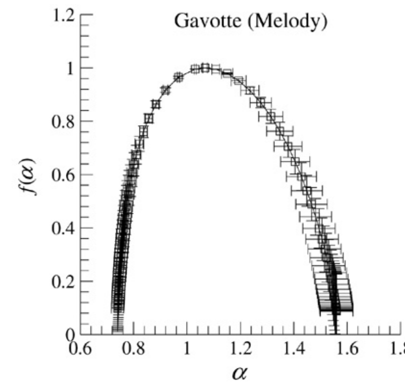

Fig. 6. Multifractal spectrum for the melody sequence of *Gavotte*. Cross marks represent the error ranges in obtaining the values of α and $f(\alpha)$.

Table 1. Among the three music pieces, *Le Cygne* has Hurst exponent H less than 0.5 both in melody and in rhythm, the corresponding α curves in Figs. 3 and 4 indeed look more rugged in profiles than the other two music pieces do. Note that the values of the Hurst exponent for all curves are not close to 1, indicating the α curves are self-affine in structure rather than self-similar.

The multifractal feature of music sequence is characterized by its spectrum $f(\alpha)$. The graph of multifractal spectrum ($f(\alpha) - \alpha$ curve) generally shows the shape of a parabola that is concave downward. The maximum of the curve occurs at $q = 0$, where $f(\alpha)$ corresponds to the box-counting dimension D_0 of the point set. For $q = 1$, $f(\alpha)$ is equal to the information dimension D_1 and the slope of the $f(\alpha)$ curve is equal to 1. The opening $(\alpha(-\infty) - \alpha(+\infty))$ of the parabola reflects the degree of irregularity in the distribution of the point set. A wide opening parabola indicates that points are not uniformly distributed along the line; rather, the tendency is to form clusters of different sizes and densities. In the special case of a monofractal, the parabola degenerates to a point.

Fig. 6 shows the multifractal spectrum curve for the melody sequence of *Gavotte*, in which cross marks denote the error ranges of α and $f(\alpha)$ values. The wide opening of the graph again indicates a non-uniform clustering structure of the sequence. Fig. 7 provides a comparison between the spectra obtained from the melodies of the three different music pieces (with error bars removed from the plot for clarity). It is observed that the opening size of each curve follows the order: *Gavotte* > *Ave Maria* > *Le Cygne*. Fig. 8 is the multifractal spectrum curve for the rhythm of *Gavotte*. Again, it is a downward opening parabola in shape, except the width of the opening is smaller than that of the melody sequence. In Fig. 9, spectra obtained from the rhythms of the three music scores are compared. It is found that the opening size of the curve

Table 2
Main fractal data derived from multifractal spectra for different music pieces

	Music scores	$\alpha(-\infty) = D_{-\infty}$	$f(0) = D_0$	$f(1) = D_1$	$\alpha(+\infty) = D_{+\infty}$	$\alpha(-\infty) - \alpha(+\infty)$
Melody	*Gavotte*	1.561	1.000	0.942	0.740	0.821
	Le Cygne	1.384	1.000	0.974	0.798	0.586
	Ave Maria	1.526	1.000	0.964	0.762	0.764
Rhythm	*Gavotte*	1.323	1.000	0.977	0.772	0.551
	Le Cygne	1.439	0.997	0.963	0.757	0.682
	Ave Maria	1.432	1.000	0.977	0.796	0.636

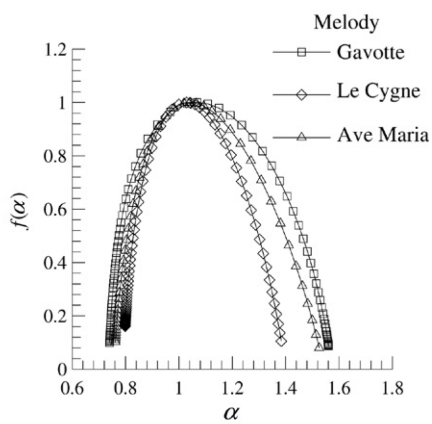

Fig. 7. Comparison of multifractal spectra for melody sequences obtained from different music scores.

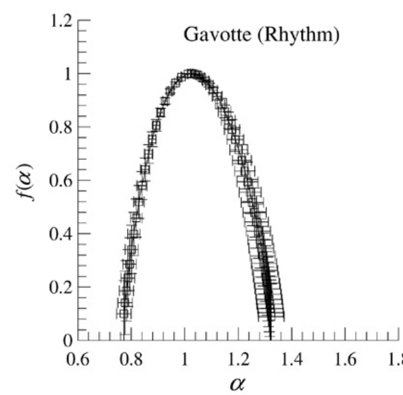

Fig. 8. Multifractal spectrum for the rhythm sequence of *Gavotte*. Cross marks represent the error ranges in obtaining the values of α and $f(\alpha)$.

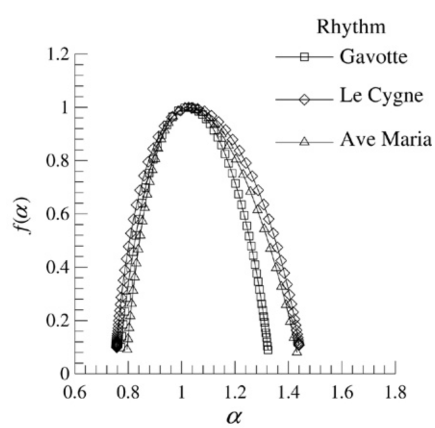

Fig. 9. Comparison of multifractal spectra for rhythm sequences obtained from different music scores.

for *Gavotte* is the smallest, while those of *Le Cygne* and *Ave Maria* are about the same. A larger opening size in the melody spectrum reflects a broader variation in pitch between notes, implying that the music may sound more bright and active. A larger opening size in the spectrum of rhythm, on the other hand, reveals more conspicuous variation in beats of the music, suggesting richer emotional traits in the expression of the music. Such differences in melody and rhythm, which are also perceivable as we actually listen to these three music works, are consistent with the results demonstrated in their multifractal spectra. Thus multifractal spectrum analysis shows a great potential to become one of the effective tools in discerning and classifying different musical styles. Various relevant main fractal data derived from multifractal spectra for different music pieces are summarized in Table 2 for further reference.

Finally, if we map the point-sequence version of the melody and rhythm of a music piece (e.g., the ones given in Fig. 1) into a two-dimensional graph with abscissa denoting the rhythmic motion and ordinate the melodic motion, a fractal curve as shown in Fig. 10 is obtained. The curve is full of many ‘plateaus’ and ‘steps’ of different sizes — a diagram similar to the ‘Devil’s Staircase’ as exhibited by the behaviors of many dynamical systems described in their parameters’ planes. The same method can be applied to diagrammatize the melody and rhythm of every music score into such a characteristic curve. Different pieces of music may vary tremendously in their characteristic curves; some may be rather smooth, others very steep and rugged. Such a mapping and the resulting characteristic curve offer yet another way to distinguish the style of a musical work. By the same token, if the behavior or geometric structure of any dynamical system shows a similar fractal curve in its parameters’ plane, attempts can also be made to map out its analogical music score using this approach.

4. Conclusion

In this paper, we have proposed a novel method to convert the melody and rhythm of a music work into distributed point sets. The fractal properties of these sequences of points can then be explored quantitatively by calculating their local Hölder exponents α and multifractal spectra $f(\alpha)-\alpha$ curves according to the multifractal formalism. Three musical pieces, *Gavotte*, *Le Cygne* and *Ave Maria* were transformed and analyzed in this study. Erratic fluctuation in the α value along the point sequence

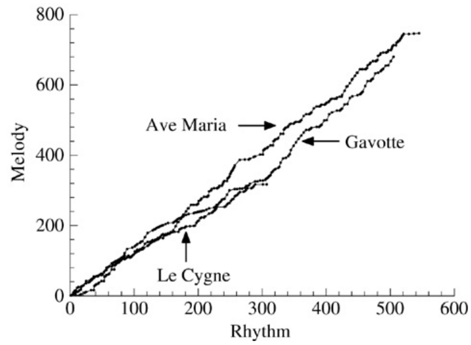

Fig. 10. Characteristic curves (Devil's Staircase) for different music pieces.

reveals how the local point distribution changes in density and hence implies a complex and irregular clustering structure in the melodic and in the rhythmic motion of music.

The multifractal spectra of the melody and rhythm sequences obtained from the three music pieces all show a familiar shape of an inverted and downward-opening parabola. A wide opening of the parabola indicates that points are not uniformly distributed along the sequence, which once again confirms the non-uniformity structure of the musical movement. Our present results show that the opening sizes of the spectra for the three melodies under study are in the following order: *Gavotte* > *Ave Maria* > *Le Cygne*; while the order of the opening sizes of the spectra for the three rhythms is: *Le Cygne* > *Ave Maria* > *Gavotte*. The physical meaning as represented by these multifractal spectra is that a relatively larger opening in the $f(\alpha)$ curves of melody and rhythm reflects the music featuring a more drastic fluctuation in pitch and richer variation in beat. Thus we are in the position to say that one piece of music is more melodious or more rhythmic than the other. The present analysis suggests that the multifractal spectrum and its relevant fractal data can be used to distinguish and classify different styles of music.

As a final point, a fractal geometry full of many 'plateaus' and 'steps' of different sizes is constructed by mapping the point sequences converted from the melody and rhythm of a music piece into a two-dimensional curve. The geometry resembles the 'Devil's Staircase' — a fractal set describing the dynamics of many interesting dynamical systems in proper parameters' planes. This fractal curve characterizes the melodic as well as the rhythmic motions of music, and is unique to each different piece of music. We have therefore provided an innovative means to disclose the intrinsic property of music.

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References

[1] B.B. Mandelbrot, The Fractal Geometry of Nature, Freeman, New York, 1983.
[2] R.F. Voss, J. Clarke, $1/f$ noise in music and speech, Nature 258 (1975) 317–318.
[3] R.F. Voss, J. Clarke, $1/f$ noise in music: Music from $1/f$ noise, J. Acoust. Soc. Am. 63 (1) (1978) 258–263.
[4] P. Campbell, Replies: Is there such a thing as fractal music? Nature 325 (1987) 766.
[5] R.F. Voss, Fractals in nature: From characterization to simulation, in: H.-O. Peitgen, D. Saupe (Eds.), The Science of Fractal Images, Springer, New York, 1988, pp. 21–70.
[6] K.J. Hsu, A.J. Hsu, Fractal geometry of music, Proc. Natl Acad. Sci. USA 87 (1990) 938–941.
[7] K.J. Hsu, A.J. Hsu, Self-similarity of the $1/f$ noise called music, Proc. Natl Acad. Sci. USA 88 (1991) 3507–3509.
[8] K.J. Hsu, Fractal geometry of music: From bird songs to Bach, in: A.J. Crilly, R.A. Earnshaw, H. Jones (Eds.), Applications of Fractals and Chaos, Springer-Verlag, New York, 1993, pp. 21–39.
[9] Y. Shi, Correlations of pitches in music, Fractals 4 (1996) 547–553.
[10] M. Bigerelle, A. Iost, Fractal dimension and classification of music, Chaos Solitons Fractals 11 (2000) 2179–2192.
[11] G. Madison, Properties of expressive variability patterns in music performances, J. New Music Res. 29 (2000) 335–356.
[12] B. Manaris, D. Vaughan, C. Wagner, J. Romero, R.B. Davis, Evolutionary music and the Zipf–Mandelbrot law: Developing fitness functions for pleasant music, Lecture Notes in Computer Science 2611 (2003) 522–534.
[13] G. Gunduz, U. Gunduz, The mathematical analysis of the structure of some songs, Physica A 357 (2005) 565–592.
[14] P. Grassberger, Generalized dimensions of strange attractors, Phys. Lett. A 97 (1983) 227–230.
[15] U. Frisch, G. Parisi, On the singularity structure of fully developed turbulence, in: M. Ghil, R. Benzi, G. Parisi (Eds.), Turbulence and Predictability in Geophysical Fluid Dynamics and Climate Dynamics, North-Holland, Amsterdam, 1985, pp. 84–88.
[16] T.C. Halsey, M.H. Jensen, L.P. Kadanoff, I. Procaccia, B.I. Shraiman, Fractal measures and their singularities: The characterization of strange sets, Phys. Rev. A 33 (1986) 1141–1151.
[17] A. Chhabra, C. Meneveau, R.V. Jensen, K.R. Sreenivasan, Direct determination of the $f(\alpha)$ singularity spectrum and its application to fully developed turbulence, Phys. Rev. A 40 (1989) 5284–5294.
[18] A. Chhabra, R.V. Jensen, Direct determination of the $f(\alpha)$ singularity spectrum, Phys. Rev. Lett. 62 (1989) 1327–1330.
[19] B.B. Mandelbrot, J.W. Van Ness, Fractional brownian motions, fractional noises and applications, SIAM Rev. 10 (4) (1968) 422–437.