Credit-Risk Term Structure of Interest Rate: Application on Put Option and Cap

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ABSTRACT

Concern with increasing bankruptcy problems in practice, the credit risk of counterparty default has been paid attention for pricing financial assets. This article bases on two Markov processes, Hull-White model and Jarrow-Turnbull model, to develop a risky term structure of interest rate. In addition, we also extend HW two-factor model to build a richer pattern of risky term structure. The risky trees not only fit the original term structure but also embrace credit information of underlying assets. Throughout lattice calculation we can derive future spot rate and the parameters of bankruptcy processes, which can be estimated from observable data. Therefore, it is a useful method for pricing credit sensitive bonds and spread-adjusted contracts, such as options of interest rate, cap, and swaps. The applications include two simulations of interest rate derivatives, put option and cap. With the numerical analysis we infer the impact of different parameter inputs on option value.

Key words: Interest rate term structure, credit risk, trinomial tree, cap, put option

1. Introduction

The transactions of interest-rate-contingent claims such as caps, swaptions and bond options have become popular these years. In the previous decades, many faculties were dedicated to the term structure of default-free interest rate for pricing interest-rate derivatives that are assumed no default risk of counter party. In the real world, however, many options and financial assets contain option-like payoffs sold by firms with limited assets or they are traded on OTC without insurance. For such situation, the no-default assumption is far less defensible. Concern with increasingly bankruptcy problems in practice, recently academic literature has taken both early default and interest rate risk into account.

1.1 Credit Risk

Merton (1974) brought up a concept about risk structure of interest rates for pricing corporate liabilities. They related "risk" as the possible gains or losses to bondholders stemmed from the changes in the probability of default not the changes in interest rates in general. He pointed out, at given term structure, the price differentials among bonds are as results of differences in the probability of default. The model is so-called structural model, which focuses on the relation between default and asset value in an explicit way. However, Merton's model does not generate the levels of yield spreads that observed in the market.

Follow the Merton's idea, more examples of imbedded options on the corporate debt have been proven, such as Black and Cox (1976), Ho and Singer (1982), Chance (1990), and Kim, Ramaswamy, and Sundaresam (1993). They took defaultable derivatives as contingent claims not on the financial securities themselves, but as "compound options" on the assets or underlying the financial securities. Nevertheless, these approaches are difficult to implement in practice because all of the firm's assets are neither tradable nor observable. Moreover, all of the other liabilities of the firm senior to the corporate debt must first (or simultaneously) be valued, that brings about time-consuming computation.

Alternative default risk factor stems from the derivatives writer. Such an option, called vulnerable option, i.e., option is privately written and is not guaranteed by a third party, is mentioned in the article of Johnson and Stultz (1987). They proved the value of a vulnerable European option could fall with time to maturity, with the interest rate, and with the variance of the underlying asset. Also, they study how the comparative-static properties of vulnerable options differ from those of options without default risk. Hull and White (1995) extended Johnson and Stultz model to cover situations where other equal ranking claims can exit. It shows how the value of a vulnerable security can be related in a consistent way to the no-default value of the security, the values of default-free zero-coupon bonds, and the values of vulnerable zero-coupon bonds that would be issued by derivative writer. However, they do not price options on assets with credit risks nor do they analyze the hedging of vulnerable options.

Jarrow and Turnbull approach (1995) presents a technique for valuing options on a term structure of securities subject to credit risk. This approach value defaultable coupon bonds and vulnerable options in the foreign currency which takes as given a stochastic term structure of default-free interest rates and a stochastic maturity specific credit-risk spread. The advantage of this article is, given risk-free term structure and risky term structure, option type features can be priced in an arbitrage free manner using the martingale measure technology. In addition, it provides a closed-form lattice result.

Nevertheless, Jarrow and Turnbull merely used BDT default-free term structure to simplify the process of the evaluation. Note that BDT model is a conventional binary tree with probabilities of 0.5, the model lacks of sufficient information about the evolution over time of the term structure of volatilities and substantial inability to handle the conditions where the impact of a second factor could be of relevance. To enhance the effectiveness of pricing model, better risk-free term structure should be concerned.

1.2 Risk-Free Term Structure of Interest Rate

Ho and Lee (1986) were pioneers in the development of no-arbitrage model in the form of a binomial tree of discount bond prices. The model involves two parameters including the short-rate standard deviation and the market price of short rate. By Ho and Lee model zero-coupon bond and European options on zero-coupon bond can be valued analytically. Nevertheless, the drawbacks of the model are no mean-reversion process and it gives the user very little flexibility in choosing the volatility structure since all spot and forward rates have the same standard deviation, i.e. the average direction in which interest rates move over the next short period of time is always the same.

Heath, Jarrow and Morton (1992) proposed a general approach to constructing models of the term structure which involves specifying the volatilities of all forward rates at all times and the initial values of the forward rates are chosen to be consistent with the initial term structure. In addition, they extended one-factor to multi-factor arbitrage-free models of the term structure. A multi-factor HJM model probably provides the most realistic description of term structure movements, however, for lattice- or finite-differences- based approaches the computational cost generally grows with the power of the number of factors, HJM model has to be implemented by using the Monte Carlo technique or a non-recombining tree. Intrinsically, HJM model is a non-Markov, which indicates the distribution of interest rates in the next period depends on the current rate and also on rates in earlier period. In that it brings about the HJM process cannot be mapped onto a recombining tree, thus compound or American options cannot be dealt with by back-wards induction using finite-difference grids or recombining lattices, and number of nodes at time grows exponentially with so that accurate pricing is computationally extremely time consuming.

A significant breakthrough took place when Hull and White (1994) introduced a class of models, i.e. the Hull-White extended-Vasicek model, to incorporate deterministically mean-reverting features. They noted, in a long run, all yield curves should become reasonably flat and a positive mean reversion must prevail in order to price very long caps. What is the comparative advantage brought about by HW approach is that allows perfect matching of an arbitrary initial term structure and analytically tractable. With recombining trinomial lattice, not only closed–form solutions could be obtained for the prices of derivatives but model calibration also could be carried out in an efficient way. In their companion sequel article (Hull, 1994), a two-factor Markov model of the term structure was proposed, which is a method for combining trinomial trees for two correlated variables into a single three-dimensional tree describing the joint evolution of the variables.

This article applies trinomial risk-free term structure and martingale probability of default to build a risky term structure. Furthermore, it also extends to a three-dimensional term structure.

2. Material and Methods

This section describes the approaching steps of building risky term structure. The basis model is Hull-White Model for which we can build a risk-free interest rate tree and then apply it with Jarrow-Turnbull model. Since both models are Markov, we can derive a combining risky tree. The specific description is as following.

2.1 Hull-White One-Factor Model

The obvious difference between interest rates and stock prices is that short-term interest rate will gradually pull back to some long-run average level. Hence the short rate should follow a mean-reverting process. See Figure 1, when the short rate is above (below) a long-term level it should experience a downward (upward) pull towards this level. The process of short rate proposed by Hull and White (1994) is described as:

$$dr = (\theta(t) - ar)dt + \sigma dz \qquad (1)$$

or

$$dr = a\left[\frac{\theta(t)}{a} - r\right]dt + \sigma dz \; . \qquad (2)$$

Where a and σ are constants. The process of mean reversion is of the short rate pulled to a level $\theta(t)/a$ at speed of a. Superimposed upon this "pull" is a normally distributed stochastic term $\sigma\, dz$. Once a and σ are chosen then the entire term structure can be determined. Upon numerical implementation, HW model constructs a computational trinomial lattice as an interest-rate tree without risk. The discrete-time tree comprises mean reversion character and assumes the discount rates vary from node to node. And if the time step on the tree is Δt, the rates on the tree are the continuously compounded Δt- period rates. A usual assumption is that the Δt-period rate, R, follows the same stochastic process as the instantaneous rate, r, in the corresponding continuous-time model.

Tree	Branching	Probability
Top Edge	$(i, j) \to (i+1, j)$ $(i, j) \to (i+1, j-1)$ $(i, j) \to (i+1, j-2)$	$\Pr(i, j, 1) = \frac{7}{6} + \frac{j^2M^2 + 3jM}{2}$ $\Pr(i, j, 0) = -\frac{1}{3} - j^2M^2 - 2jM$ $\Pr(i, j, -1) = \frac{1}{6} + \frac{j^2M^2 + jM}{2}$
Middle	$(i, j) \to (i+1, j+1)$ $(i, j) \to (i+1, j)$ $(i, j) \to (i+1, j-1)$	$\Pr(i, j, 1) = \frac{1}{6} + \frac{j^2M^2 + jM}{2}$ $\Pr(i, j, 0) = \frac{2}{3} + j^2M^2$ $\Pr(i, j, -1) = \frac{1}{6} + \frac{j^2M^2 - jM}{2}$
Bottom Edge	$(i, j) \to (i+1, j+2)$ $(i, j) \to (i+1, j+1)$ $(i, j) \to (i+1, j)$	$\Pr(i, j, 1) = \frac{1}{6} + \frac{j^2M^2 - jM}{2}$ $\Pr(i, j, 0) = -\frac{1}{3} - j^2M^2 + 2jM$ $\Pr(i, j, -1) = \frac{7}{6} + \frac{j^2M^2 - 3jM}{2}$

Figure 1 The Branching and Probability in Each Region

The first step for constructing r-tree involves defining a new variable r^* and sets $\theta(t)$ and the initial value of r^* equal to zero. Equation (3) is the process for r^*, and its mean and the variance of the change in r^* are given by (4) and (5):

$$dr^* = -ar^* dt + \sigma_r dz_r \,. \tag{3}$$

$$E[dr^*] = \mathrm{M}x^* = (e^{-a\Delta t} - 1)x^* \tag{4}$$

$$Var\,[dr^*] = \mathrm{V} = \sigma^2 (1 - e^{-2a\Delta t})/2a \tag{5}$$

In addition, the space between r-values on the trinomial lattice equals $\sqrt{3\mathrm{V}}$ and the non-standard branching takes place at $\pm \max j$ where $\max j$ is the smallest integer of greater than $0.184/\mathrm{M}$. At node $j\Delta r^*$, the up-, middle-, and down-branching probabilities are as following. Using recursively computation the price of zero-coupon bond at each time step, $(i+1)\Delta t$, is given by

$$P((i+1)\Delta t) = \sum_{-n(i)}^{n(i)} Q(i,j) \exp[-(alpha(i) + j\Delta r)\Delta t] \tag{6}$$

$$Q(i,j) = \sum_k Q(i-1,k) * \Pr(k,j) \exp[-(alpha(i-1) + k\Delta r)\Delta t] \tag{7}$$

$$alpha(i) = \frac{\ln \sum_{-n(i)}^{n(i)} Q(i,j) \exp(-j\Delta r \Delta t) - \ln P(i+1)}{\Delta t} \tag{8}$$

with $Q(0,0) = 1$ and $r(0,0) \equiv alpha(0)$. Where $Q(i,j)$ denotes the value today of a certain, without default risk, one dollar paid in state j at time i, $\Pr(k,j)$ is the probability of moving from time-i node k to node j at time $(i+1)$. Add $alpha(i)$ to r^* interest rate tree then the final interest rate tree matches the initial term structure. Since computational lattices are discrete but finite, numbers of zero-coupon bonds are assumed to describe the default-free term structure and the short rates are also regarded as the driven factor for pricing interest derivatives.

2.2 Jarrow-Turnbull Model

The above bond process for default-free debt is assumed that pricing process depends only on the spot interest rate. However, when we consider the credit risk embedded in the underlying asset, the branching probabilities of each node would not be the same. These so-called pseudo- probabilities need to be adjusted when default occurs. Jarrow and Turnbull (1995) use BDT model, which is a short-term interest rate lognormal distribution with simple branching probability of 0.5, to handle risky bonds.

Instead of BDT model, we select Hull and White's default-free term structure model as a better approach for applying in the JT model since it takes more consideration about mean-reversion and a good fit of initial term structure.

Assume spot interest rate process and the bankruptcy process are independent under the pseudo-probabilities we use the observed term structures of zero-coupon bond price and risky corporate bond to determine the martingale probabilities of default. Let $\upsilon(t,T;DS_t)$ is the time-t value of a zero-coupon bond issued by the credit risk firm, and the symbol DS_t denotes the default status of the contract at time t as follows:

$$DS_t \equiv \begin{cases} \overline{D}\,; \text{Default has not occurred at time } t \\ D\,; \text{Default has occurred at or before time } t \end{cases} \tag{9}$$

If default occurs, the buyer will receive less than the promised amount, i.e. a recovery rate of δ or a fraction amount of principle. For example, the risky bond's value at the maturity is

$$\upsilon(T,T;DS_t) = \text{Face value} * \begin{cases} 1;\ \text{probability}\ \ 1-\text{u}(t) & \text{if}\ DS_t = \overline{D} \\ \delta;\ \text{probability}\ \ \text{u}(t) & \text{if}\ DS_t = D \end{cases} \tag{10}$$

Where $\text{u}(t)$ is time-t martingale probability that we expect default may occur at next time $t+1$, however, it has not happened yet. Thus, there are two possible credit status, default or non-default, at every interest rate level of $r(i, j)$. On the other hand, if default has occurred at or before time t, the risky bond still remains in default in the future and the payoff ratio of maturity is δ. Figure 2 shows a two-period risky interest tree, rectangular shape denotes default has not occurred till now and ellipse shape denotes default does occur. This tree simultaneously considers interest rate movement and default probabilities. This model assumes both risk-free process and default process are independent, where credit risk is embedded in the value of coupon bond.

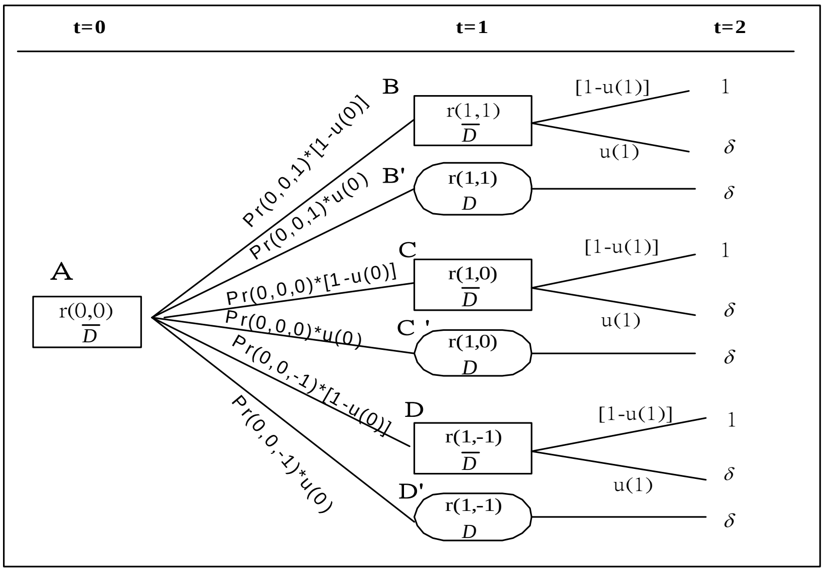

Figure 2 Two Period Risky Debt Default Process

The values of six possible statuses at time t=1 for a risky bond with maturity of two are:

$$\begin{aligned}
&\upsilon_B\ (1,2;\overline{D}) = 100 \times \{[1-\text{u}(1)] + \text{u}(1)\delta\} \times \exp(-r(1,1)) \\
&\upsilon_{B'}\ (1,2;D) = 100 \times \delta \times \exp(-r(1,1) \\
&\upsilon_C\ (1,2;\overline{D}) = 100 \times \{[1-\text{u}(1)] + \text{u}(1)\delta\} \times \exp(-r(1,0)) \\
&\upsilon_{C'}\ (1,2;D) = 100 \times \delta \times \exp(-r(1,0)) \\
&\upsilon_D\ (1,2;\overline{D}) = 100 \times \{[1-\text{u}(1)] + \text{u}(1)\delta\} \times \exp(-r(1,-1)) \\
&\upsilon_{D'}\ (1,2;D) = 100 \times \delta \times \exp(-r(1,-1))
\end{aligned} \tag{11}$$

The two-year risky tree is derived from combination of Hull-White trinomial tree and Jarow-Turnbull martingale probabilities. If the default has not occurred at time t then the conditional (martingale) probability that default occurs at time $t+1$ is denoted by $\text{u}(t)$, otherwise the martingale probability will be $1-\text{u}(t)$ in the no-default condition. Under the martingale probabilities (see Jarrow and Turnbull 1995), the value of normalized prices with only one period zero-coupon bond is given by

$$\frac{\upsilon(0,1;\overline{D})}{A(0)} = E_t^Q\left[\frac{\upsilon(t,t;DS_t)}{A(t)}\right]. \tag{12}$$

Where $A(t)$ is discount factor at time t, i.e. $A(1) = \exp(r(0))$ and $A(0) = 1$, and E_t^Q denotes the time-t conditional expected value when we are computing the expected value of the option at time $t+1$, using the equivalent martingale probabilities given whatever information is available at time t. Simply equation (11), the formula of risky bond value is

$$\begin{aligned}\upsilon(0,1,\overline{D}) &= 100\times\left\{[1-\mathrm{u}(0)]+\delta\times\mathrm{u}(0)\right\}/A(1)\\ &= \mathrm{BF}(0,1)\times\left\{[1-\mathrm{u}(0)]+\delta\times\mathrm{u}(0)\right\}\end{aligned} \tag{13}$$

Note that $BF(0,t)=\dfrac{100}{A(t)}$. The martingale probability that default occurs by the end of the first year can be derive as

$$\mu(0)=\left[1-\frac{\upsilon(0,1,\overline{D})}{\mathrm{BF}(0,1)}\right]\Big/(1-\delta). \tag{14}$$

Using long-term maturity data of asset we can extend long-run default process. Assume a risky corporate bond matures at second year. The present value of default-free bond with maturity time two, which denotes as $\mathrm{BF}(0,2)$, is

$$\mathrm{BF}(0,2)=\begin{Bmatrix}\mathrm{Pr}(0,0,1)\times\exp(-\mathrm{r}(1,1))+\\ \mathrm{Pr}(0,0,0)\times\exp(\text{-}\mathrm{r}(1,0))+\\ \mathrm{Pr}(0,0,\text{-}1)\times\exp(\text{-}\mathrm{r}(1,\text{-}1))\end{Bmatrix}\times\exp(-r(0,0)) \tag{15}$$

Base on the assumption of independence between the branching probability of risk free tree and martingale probability of default risk, the calculation process for pricing the risky bond is as equation (16).

$$\upsilon(0,2;\overline{D})=100\times\begin{Bmatrix}\mathrm{Pr}(0,0,1)\times[1\text{-}\mathrm{u}(0)]\times\upsilon_{\mathrm{B}}(1,2;\overline{D})+\mathrm{Pr}(0,0,1)\times\mathrm{u}(0)\times\upsilon_{\mathrm{B'}}(1,2;D)+\\ \mathrm{Pr}(0,0,0)\times[1\text{-}\mathrm{u}(0)]\times\upsilon_{\mathrm{C}}(1,2;\overline{D})+\mathrm{Pr}(0,0,1)\times\mathrm{u}(0)\times\upsilon_{\mathrm{C'}}(1,2;D)+\\ \mathrm{Pr}(0,0,\text{-}1)\times[1\text{-}\mathrm{u}(0)]\times\upsilon_{\mathrm{D}}(1,2;\overline{D})+\mathrm{Pr}(0,0,\text{-}1)\times\mathrm{u}(0)\times\upsilon_{\mathrm{D'}}(1,2;D)\end{Bmatrix}\times\exp(-r(0,0)) \tag{16}$$

Use equation (11), (14), (15), equation (16) can be in short as

$$\upsilon(0,2;\overline{D})=\mathrm{BF}(0,2)\times\left\{\mathrm{u}(0)\times\delta+[1\text{-}\mu(0)]\times\{[1-\mu(1)]+\mathrm{u}(1)\delta\}\right\} \tag{17}$$

Transform equation (17), the first year default probability is

$$\mu(1)=\left\{1-\left[\frac{\upsilon(0,2,\overline{D})}{\mathrm{BF}(0,2)}-\mathrm{u}(0)\delta\right]\Big/[1-\mathrm{u}(0)]\right\}\Big/(1-\delta) \tag{18}$$

By similar methodology, we could derive other default probabilities, for example, the expected default probability at second year is

$$\mu(2)=\left\{1-\left\{\left[\frac{\upsilon(0,3;\overline{D})}{\mathrm{BF}(0,3)}-\mathrm{u}(0)\delta\right]\Big/[1-\mathrm{u}(0)\delta]\;-\mathrm{u}(1)\delta\right\}\Big/[1\text{-}\mathrm{u}(1)]\right\}\Big/(1-\delta) \tag{19}$$

As soon as whole period default probabilities have been derived, synchronize each possible future spot rate with above martingale default process, the pattern of combined risky tree is completed as Figure 3. If default happens, which denotes as the symbol of D on the lattice, we only discount default value by default-free interest rate since we have consider credit risk in the term structure. However, if default doesn't happen yet, denotes as the symbol of $\overline{D}$, time-t discount value is the weighted average of both payments of go default and not being default at time $t+1$.

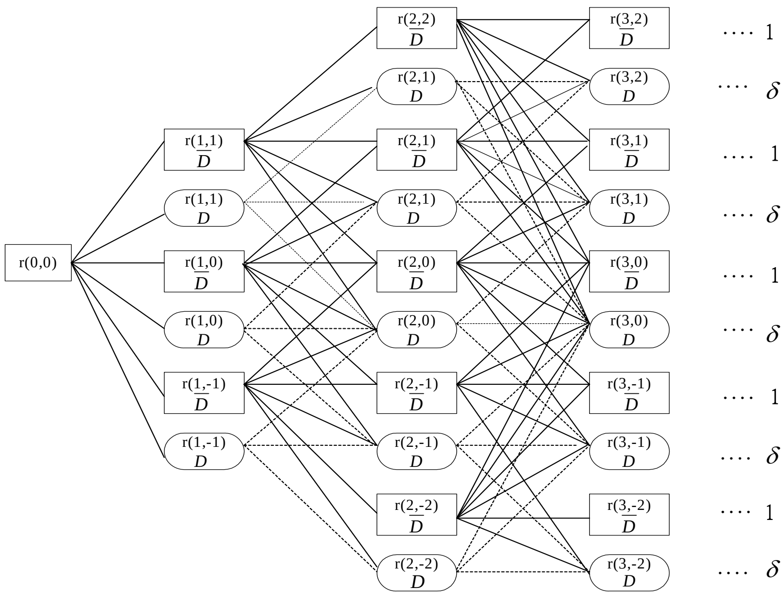

Figure 3 Risky Interest Rate Tree

Rolling back the payment at maturity we can derive the present value of risky zero bond. The specific computation process for each status value on the interest rate lattice is as followings: If default does not happen yet, then bond value is the weighted average of default and non-default payments.

$$
\begin{aligned}
\upsilon(i,T,\overline{D}_t) = & \sum_{j=-n}^{n} \Pr(i,j,k)\times(1-\mathrm{u}(i))\times\upsilon(i+1,T,\overline{D}_{t+1})\times\exp(-r(i,j)\times\Delta t) \\
& + \sum_{j=-n}^{n} \Pr(i,j,k)\times \mathrm{u}(\mathrm{i})\times\upsilon(i+1,T,D_{t+1})\times\exp(-r(i,j)\times\Delta t)
\end{aligned} \tag{20}
$$

If default has happened then bond value is only discount value of default payment.

$$\upsilon(i,T,D_t) = \sum_{j=-n}^{n}\Pr(i,j,k)\times\upsilon\left(i+1,T,D_{t+1}\right)\times\exp(-r(i,j)\times\Delta t) \tag{21}$$

The advantage of the combined risky tree is that we can use lattice process to value interest rate derivatives, such as cap and bond option, where underlying asset subject to credit risk. Furthermore, the risky lattice can be easily extended to a two-factor term structure including short rate and long rate, which is discussed in section C.

2.3 Risky Three-Dimension Term Structure

The idea of risky term structure could be extended for allowing one more factor of interest rate, which embraces future volatility patterns. Hull and White (1994) proposed a two-factor formulation of their Extended Vasicek model, per se, which is a Morkov model and could be implemented as a combining tree. The joint dynamics of short rate, r, and long rate, u, are of the form:

$$dr = \left[\theta(t) + u(t) - ar(t)\right]dt + \sigma_1 dz_1 \tag{22}$$

$$du = -budt + \sigma_2 dz_2 \tag{23}$$

Assuming $a \neq b$. The equation is similar to Hull and White (1994) one-factor model. In addition to $\theta(t)$ there is a stochastic function of $u(t)$ in the drift of r, which takes long-run interest rate trend into consideration for a reasonable description of real interest rate market. The reversion level, $u(t)$, also reverts to a level of zero at speed of b. Let $a \neq b$, $E(dz_1, dz_2) = \rho dt$ and $u(0) = 0$.

This model allows a close-form solution for pricing discount bond. In order to eliminate the dependence of the first stochastic variable on the second, u, the new variable y is introduced as

$$y = r + \frac{u}{b-a} \tag{24}$$

So that the new stochastic differential equations become

$$dy = [\theta(t) - ay]dt + \sigma_y dz_y \tag{25}$$

$$du = -budt + \sigma_u dz_u \tag{26}$$

with σ_y^2 and correlation between u and y are given by

$$\sigma_y^2 = \sigma_r^2 + \frac{\sigma_u^2}{(b-a)^2} + \frac{2\rho\sigma_r\sigma_u}{b-a} \tag{27}$$

$$E[dz_2, dz_3] = \frac{\rho\sigma_r + \sigma_u/(b-a)}{\sigma_y} \tag{28}$$

Where dz_y as dz_r and dz_2 is a Wiener process. Each of short rate and long could be mapped as a one-factor trinomial interest rate tree separately. Combine y-tree and u-tree on the assumption of zero correlation. The result is a three-dimensional tree where nine branches emanate from each status and each branching probability from $[y(i, j_y), u(i, j_u)]$ to the nine reachable nodes is the product of the unconditional probabilities associated with the trinomial tree of y and the trinomial tree of u. That is

$$\Pr(r(i, j_y, j_u)) = \Pr(y(i, j_y)) * \Pr(u(i, j_u)) \tag{29}$$

The short rate, r, is given by the initial tree to be

$$r^*(i, j_y, j_u) = j_y \Delta y - \frac{j_u \Delta u}{b-a} \tag{30}$$

$$r(i, j_y, j_u) = r^*(i, j_y, j_u) + alpha(i) \tag{31}$$

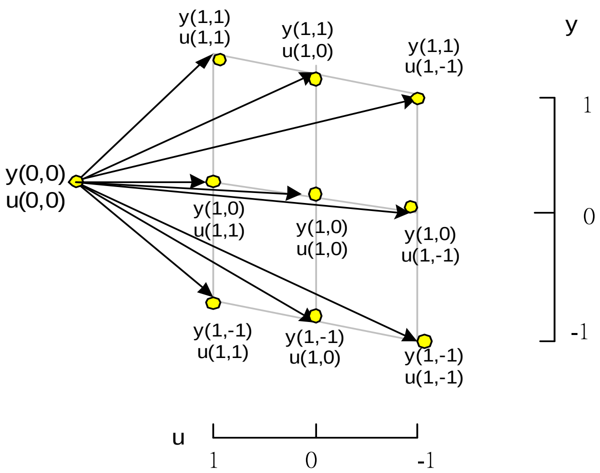

Figure 4 Hull and White Two-Factor Interest Rate Tree

Where the computations of $alpha(t)$ are as following recursively equation.

$$\mathrm{P}_{t+1} = \sum_{j_y} \sum_{j_u} \mathrm{Q}_{t,j_y,j_u} \exp\left\{-\left(alpha(t) + j_y \Delta y - \frac{j_u \Delta u}{b-a}\right)\Delta t\right\} \tag{32}$$

$$alpha(t) = \frac{\log \sum_{j_y} \sum_{j_u} \mathrm{Q}_{t,j_y,j_u} \exp\{-(j_y \Delta y - j_u \Delta u/(b-a))\Delta t\} - \log \mathrm{P}_{t+1}}{\Delta t} \tag{33}$$

$$Q_{t+1,j_y,j_u} = \sum_{j_y} \sum_{j_u^*} Q_{t,j_y^*,j_u^*} * \Pr(i, j_y, j_u) \times \exp\left\{ -\left[alpha(t) + j_y^* \Delta y - j_u^* \big/ (b-a) \right] \Delta t \right\} \tag{34}$$

Let $alpha(0) = r(0)$ to match initial term structure. The number of nodes of three-dimension interest tree is the product of two estimations of y tree and u tree. Define Q_{i,j_y,j_u} as the present value of a security that pays off \$1 when $y = \alpha_i + j_y \Delta y$ and $u = j_u \Delta u$ at time $i\Delta t$ or zero otherwise. $Q_{0,0,0}$ = 1. As Hull(1994) shows, see equation (30), the interest rate pattern could be derived after adding each time $apha(t)$ to original three-dimension tree, r^*.

Note that the risk-free interest rate tree is independent with default process. Therefore, it is easy to extend a multi-factor model. The method for combining both of two-factor interest rate tree and default process is similarly as one-factor approach but it is more complicated. The risk-free-two-factor term structure can be derived from equation (29) to (34). However, given constant recovery rate, yearly default probability, $\mathrm{u}(i)$, is the same with one-factor risky interest rate tree since the ratio of risk-free bond and corporate bond is fixed. Consider two-factor interest rate model and default process, the calculation approach for risky bond value is as followings:

If default does not happen at time t, denotes as $\overline{D}_t$, then the time t value of risky bond includes both of default and non-default payments from time $t+1$.

$$\begin{aligned} &\upsilon(i,T,\overline{D}_t) \\ &= \sum_{j_y=-n_y}^{n_y} \sum_{j_u=-n_u}^{n_u} \Pr(i, j_y, j_u, k_y, k_u) \times (1-\mathrm{u}(i)) \times \upsilon(i+1,T,\overline{D}_{i+1}) \times \exp(-r(i, j_y, j_u) \times \Delta t) \\ &\quad + \sum_{j_y=-n_y}^{n_y} \sum_{j_u=-n_u}^{n_u} \Pr(i, j_y, j_u, k_y, k_u) \times \mathrm{u}(i) \times \upsilon(i+1,T,D_{t+1}) \times \exp(-r(i, j_y, j_u) \times \Delta t) \end{aligned} \tag{35}$$

If default has happened at time t, denotes as D_t, then it remains default next period.

$$\upsilon(i,T,D_t) = \sum_{j_y=-n_y}^{n_y} \sum_{j_u=-n_u}^{n_u} \Pr(i, j_y, j_u, k_y, k_u) \times \upsilon(i+1,T,D_{t+1}) \times \exp(-r(i, j_y, j_u) \times \Delta t) \tag{36}$$

Where n_y (n_u) is the max j_y (max j_u) and k_y (k_u) is the next-period state of j_y(j_u). Note that credit-risk factor has been considered in the risky tree, the discount factors are risk free interest rates derived from Hull-White model. We use computer programming to deal with these computations.

3. Results and Discussion

This section discusses the application of defaultable tree on interest rate derivatives and presents two examples of interest rate options, put option of coupon bond and cap. By the example process, we have a thorough understanding about the approach of tree combination model. Furthermore, we examine the variation of option values in different pairs of parameters and discuss our pricing result. The data of risk-free bond is derived from government bond prices and the risky bond is derived from Far-East-Textile corporate bond.

By Hull and White (1994), given a =0.1, σ =0.01 and dt =1, the risk free interest rate tree is derived as specified in Table 1 and Figure 5. Note that the pattern of Alpha is derived from zero-coupon bond and matches original term structure. In addition, the nodes at time t are the possible movement from time $t-1$ node. The number of j at each side of Alpha pattern equals 2. The value of node-j at time t is time $(t+1)$ expect discount value of nodes $j \pm 1$. All of nodes in Figure 5 are possible future interest rats. In addition, the central line, designated as Alpha Path, matches the initial yield curve and the volatility and convergence of the tree are determined by both sizes of mean reversion, a, and volatility of interest rate, σ.

Table 1 Zero-Coupon Bond Price

Maturity / Year	Risk-Free Discount Bond	Zero Rate (%)	Forward Rate (%)	Alpha (%)
0	1.0000	0.0610	0.0610	0.0637
1	0.9383	0.0637	0.0654	0.0651
2	0.8791	0.0644	0.0655	0.0658
3	0.8233	0.0648	0.0659	0.0663
4	0.7707	0.0651	0.0655	0.0651
5	0.7225	0.0650	0.0640	0.0640
6	0.6783	0.0647	0.0635	0.0650
7	0.6362	0.0646	0.0657	0.0691
8	0.5945	0.0650	0.0710	0.0764
9	0.5516	0.0661	0.0764	0.0799

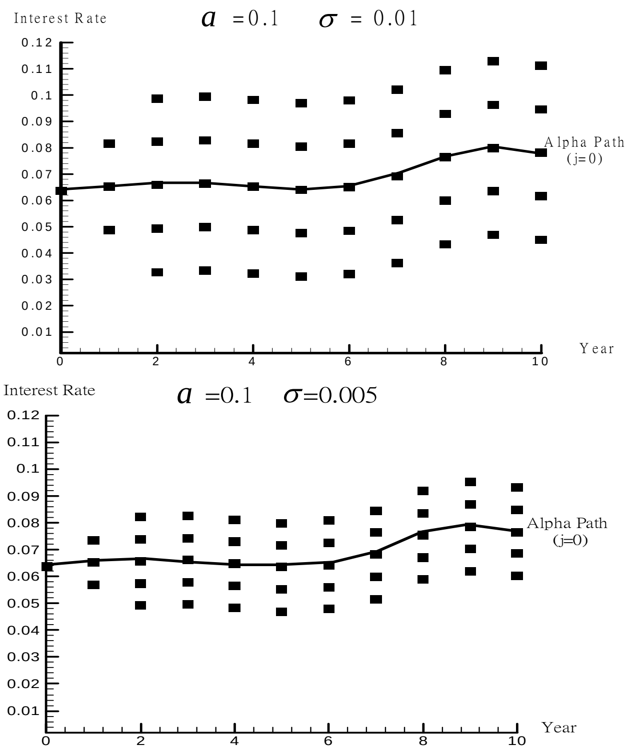

Figure 5 Hull and White One-Factor Risk Free Pattern

As illustration in Figure 5, we can see that higher volatility and less mean reversion speed would enlarge the scope of interest rate level and raise the value of interest derivatives but keeping bond value unchanged. The advantage of interest rate lattice is that favorable for discrete-time calculation and the pattern of term structure could be obtained.

Table 2 Yearly Interest Rates

t = 0	t = 1	t = 2	t = 3	t = 4	t = 5	t = 6	t = 7	t = 8	t = 9
		0.0987	0.0993	0.0981	0.0970	0.0980	0.1020	0.1094	0.1128
	0.0816	0.0823	0.0828	0.0816	0.0805	0.0815	0.0856	0.0929	0.0963
0.0637	0.0651	0.0658	0.0663	0.0651	0.0640	0.0650	0.0691	0.0764	0.0799
	0.0487	0.0493	0.0498	0.0487	0.0475	0.0485	0.0526	0.0599	0.0634
		0.0328	0.0334	0.0322	0.0310	0.0320	0.0361	0.0434	0.0469

Parameters: $a = 0.1$, $\sigma = 0.01$, *bond maturity =10 years*

Table 3 Default Probabilities for Risky Corporate Bond

Maturity / year	Risk-free Discount Bond	Corporate Bond Value with Similar Credit Class	Default Probability u(i)
0	1.0000	99.5463	0.0124
1	0.9383	93.0169	0.0319
2	0.8791	85.2143	0.0392
3	0.8233	77.6417	0.0328
4	0.7707	71.0558	0.0349
5	0.7225	65.0404	0.0493
6	0.6783	59.0501	0.0376
7	0.6362	54.0254	0.0441
8	0.5945	49.0452	0.0414
9	0.5516	44.3058	0.0722

Other parameter: $\delta = 0.3$

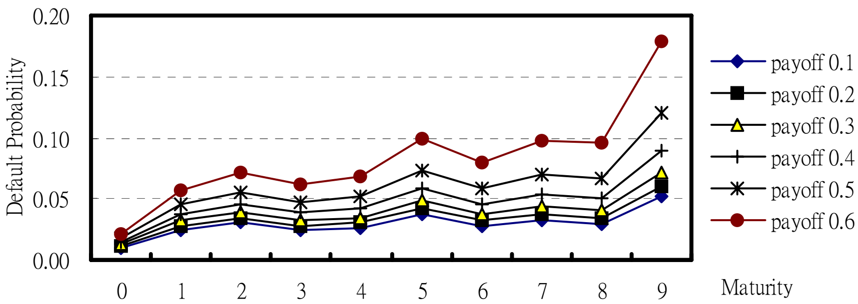

Figure 6 Credit Risk Curve

The simple branching nodes of interest rates are in Table 2. Using these data we can price risk-free financial assets or the value of embedded options. Nevertheless, consider the counterparty-default risk we further extend to credit process. By the Jarrow-Turnbull model, we could see that default probability is a function of the ratio of Treasury bond to risky bond prices. Using both data of risk free zero bond and risky zero bond, we could compute yearly default probabilities as Table 3.

Since the value of risk free bond is higher than risky bond each year that implies there is credit risk embedded in the corporate bond, in particularly, the payment at maturity is uncertain, investor likely only receive a fraction of principle of the bond. Each year, the bond has different default probability those expresses a certain credit status of the bond. If default does not happen in the previous time, there is still a default risk for the future payment. Moreover, as Figure 6 shows that longer-term risky corporate bonds are more uncertain than short-term risky bond, thus the default probability rises as the maturity of risky bond increases.

Since each time default risk is not the same, the lattice of risk term structure provides full credit information and is a better implementation for pricing interest rate derivatives subject to credit risk. The present value of derivatives is obtained by rolling backward from the maturity to time zero step by step. Using above information, two examples of interest derivatives are represented as follows.

3.1 Put Option Example

A put option matures at fifth year with strike price of 0.55 (unit of one dollar). The underlying asset of the put is the risky corporate bond we have mentioned above. Assume a bondholder buys a put on the risky bond for hedging against value decline of the bond in the expectation of rising interest rate in market. With the put option, conservative bondholder places a limitation of minimum revenue, thus if the bond price is less than the strike price, the hedger can receive the interest rate spread.

See Table 4, fifth-year values of non-default and default bonds are specified as $\upsilon_{j,\bar{D}}$ and $\upsilon_{j,D}$ respectively, and nearby column is the payoff of the European put option. With the risky interest rate tree we can derive several values of risky bond, including default and non-default status at fifth-year.

Table 4 Bond Price and the Payoff of Put Option at Fifth Year

Status of $\upsilon_{j,DS}$	Risky Bond Value at 5_{th} Year	Payoff of Put Option
$\upsilon_{2,\bar{D}}$	0.51761	0.0324
$\upsilon_{2,D}$	0.18387	0.3661
$\upsilon_{1,\bar{D}}$	0.55411	0
$\upsilon_{1,D}$	0.19683	0.3532
$\upsilon_{0,\bar{D}}$	0.59321	0
$\upsilon_{0,D}$	0.21072	0.3393
$\upsilon_{-1,\bar{D}}$	0.63507	0
$\upsilon_{-1,D}$	0.22559	0.3244
$\upsilon_{-2,\bar{D}}$	0.67986	0
$\upsilon_{-2,D}$	0.24150	0.3085

$\upsilon_{j,\bar{D}}$ is non-default value of bond at the interest rate level of $R_{i,j}$, on the contrary, $\upsilon_{j,D}$ is value of bond where default has happened. The recover rate of the bond=0.3 , strike price =0.55, mean reversion=0.1, and volitility=0.01.

With put privilege bondholder could receive payment of interest spread from financial institution to assure the minimum profit of bond investment. The payoff of put option equals as :

$$\text{Payoff of Put} = \max(\text{strike price} - \upsilon_{j,DS}\ ,\ 0) \tag{37}$$

Rolling back the risky tree we could derive present value of put option. The result and the impact of other parameter inputs for option value are as follows.

Table 5 Simulation of Put Value for Different Inputs of a and σ

volitility	mean reversion				
	0.1	0.2	0.3	0.4	0.5
0.01	0.0365	0.0353	0.0348	0.0348	0.0348
0.02	0.0451	0.0399	0.0365	0.0348	0.0348
0.03	0.0533	0.0457	0.0421	0.0386	0.0366
0.04	0.0607	0.0511	0.0473	0.0423	0.0394
0.05	0.0726	0.0559	0.0522	0.0459	0.0421

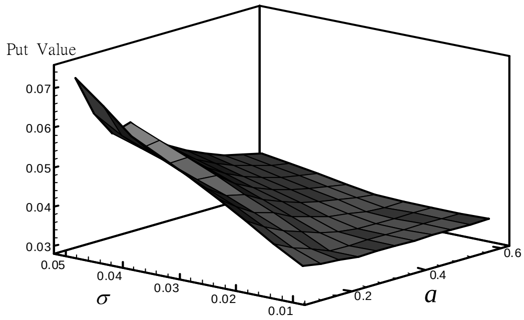

Figure 7 Graph for Different Inputs of a and σ

Table 5 shows simulation of put value among different pairs of parameters of a and σ. The graph in Figure 7 implies that the slower mean reversion and higher volatility, the more valuable put option. Although mean reversion, deviation of interest and default probabilities determine the pattern of the lattice of risky interest rate. However, strike price and the recovery payoff are also decisive factors of cash flow and could be made changes though negotiation with option writers. Thus, we simulate both of them next.

Table 6 Simulation of Different Inputs of Strike and Recovery Rate

strike price	recovery rate						
	0.1	0.2	0.3	0.4	0.5	0.6	0.7
0.3	0.0184	0.0143	0.0091	0.0024	0.0000	0.0000	0.0000
0.35	0.0224	0.0188	0.0142	0.0081	0.0012	0.0000	0.0000
0.4	0.0264	0.0233	0.0194	0.0141	0.0068	0.0007	0.0000
0.45	0.0304	0.0278	0.0245	0.0201	0.0140	0.0054	0.0004
0.5	0.0344	0.0323	0.0297	0.0262	0.0212	0.0138	0.0047
0.55	0.0400	0.0385	0.0365	0.0340	0.0307	0.0261	0.0187
0.6	0.0532	0.0526	0.0519	0.0510	0.0497	0.0478	0.0445
0.65	0.0805	0.0804	0.0802	0.0799	0.0796	0.0790	0.0784
0.7	0.1145	0.1145	0.1145	0.1145	0.1145	0.1145	0.1145

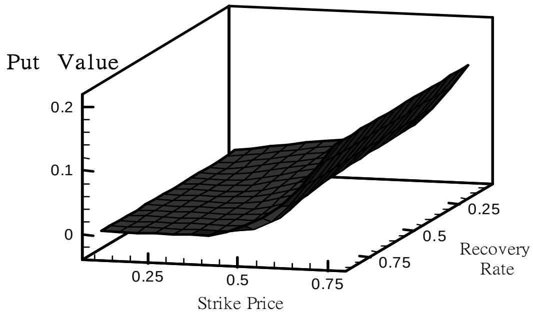

Figure 8 Graph for Different Inputs of Strike and Recovery Rate

Table 6 and Figure 8 show the impact of different inputs of strike price and recovery rate on put value. Recovery rate has little influence on put value. Even though the guarantee of higher recovery rate assures higher refund payment, however it also brings up higher imbedded default probabilities among similar credit-risk bonds, as the illustration in Figure 6. In equilibrium, the value of risky bond doesn’t follow the change of recovery rate.

Table 7 Two Factor Results

b	a				
	1.0000	2.0000	3.0000	4.0000	5.0000
0.1	0.0407	0.0356	0.0348	0.0348	0.0348
0.2	0.0375	0.0350	0.0348	0.0348	0.0348
0.3	0.0357	0.0348	0.0348	0.0348	0.0348
0.4	0.0353	0.0348	0.0348	0.0348	0.0348
0.5	0.0351	0.0348	0.0348	0.0348	0.0348

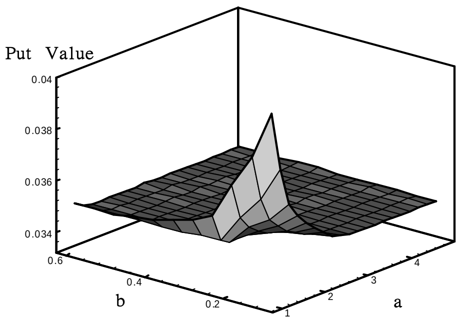

Figure 9 Graph for two-factor simulation

On the other hand, strike price is an important factor of pricing put value. The maturity payoff of the put is that risky bond value deducted from strike price, hence higher strike price would increases put value.

Throughout the simulation of two-factor risky tree we could obtain put option value in Table 7. Since two-factor model considers the impact of mean reversion and volatility on both of short rate and long rate simultaneously, it provides a richer pattern of term structure movements and accurate valuation of interest rate derivatives than one-factor model. As Figure 9 shows, the lower mean reversion speed of interest rate, the higher value of put option. On the other hand, stronger mean reversion, which goes along with a more convergent tree, depresses the value of long-dated options. Moreover, volatilities of short and long rate also positively contribute to the value of put option, which is the same with the reference of Black-Scholes model. The following example is a simulation of a cap and it illustrates the application of interest rate tree for calculation duration cash flows of interest payment.

3.2 Cap Example

A company issues a floating-rate coupon bond for which the interest rate is reset every year. The reset rate depends on prevailing one-year spot interest rate plus 3 bp. However, a contingent claim of 6% cap rate has also been embedded in the contract during total life of the bond for limiting the risk of increasing interest cost. Upon the cap rate, when floating interest rate is greater than cap rate, interest payment investors receive is only cap rate applied to the principal of contract and investor can't receive the benefit of advanced interest rate. On the other hand, if future interest rate is less than cap rate, the company still pays the interest base on floating interest rate at the end of year.

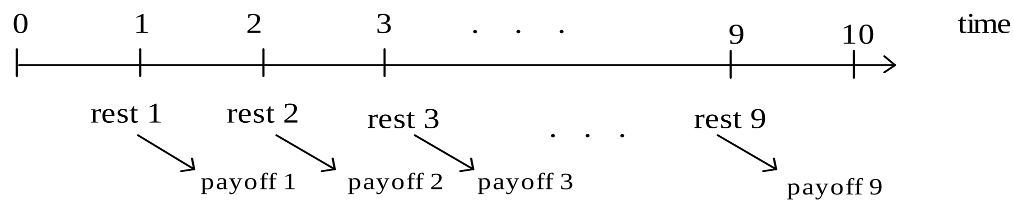

Figure 10 Payment and Reset Dates

In terms of hedging capital cost, the cap ensures that the effective interest rate paid on the debt is capped at a fixed rate. In this case, see Figure 10, with ten-year corporate there are a total of 9 reset dates (at times 1, 2, …, 9 years) and 9 payoffs from the caps (at times 2, 3, …, 10 years). The risk-free interest rate tree we have derived prior section could be employed as future spot rate. Therefore, a cap, as seen as a combination of caplets, is a series interest-rate call options. The time-t value of a caplet, which leads to a payoff at time $t+1$, is given by:

$$caplet(t)\text{ value} = \begin{cases} \dfrac{L\tau}{1+R_t\tau}\max(R_t - X\,,\ 0) & ,\ \text{if time } t+1 \text{ does not default} \\ 0 & ,\ \text{if time } t+1 \text{ default} \end{cases} \tag{38}$$

Where L is the principal of the option and τ is the tenor, a period between reset, which we assume is one year. The sign R_t is time-t spot rate, and X is the cap rate. The cash flow of risky bond includes both of payments of floating, but limited, interest rate and the payment of risky maturity principle. In addition, yearly interest rate payment is not affected by the credit risk which means bond holders still receive interest payment in the circumstance of default. Moreover, the cap is identical to that investor sells an interest rate call option to the bond issuer; therefore we need deduct value of embedded option price from the value of risky bond and derived net value of bond. Table 8 and Figure 11 show the simulation of cap value, which depends on the size of volatility of interest rate and strike price of option. As we mention above, with less mean reversion and larger volatility there is a larger scope of interest tree, therefore, the value of cap is higher.

Table 8 Cap Value in One-Factor Model

cape rate	volatility 0.01	0.02	0.03	0.04	0.05	0.06	0.07
0.02	0.0660	0.0668	0.0671	0.0678	0.0681	0.0682	0.0703
0.03	0.0572	0.0575	0.0578	0.0582	0.0588	0.0610	0.0632
0.04	0.0484	0.0484	0.0491	0.0494	0.0516	0.0538	0.0560
0.05	0.0353	0.0376	0.0399	0.0422	0.0444	0.0466	0.0489
0.06	0.0280	0.0303	0.0326	0.0349	0.0372	0.0395	0.0417
0.07	0.0060	0.0083	0.0105	0.0126	0.0146	0.0167	0.0186
0.08	0.0046	0.0068	0.0091	0.0112	0.0133	0.0153	0.0173
0.09	0.0000	0.0054	0.0077	0.0098	0.0120	0.0140	0.0160
0.1	0.0000	0.0000	0.0063	0.0085	0.0106	0.0127	0.0147
0.11	0.0000	0.0000	0.0049	0.0071	0.0093	0.0114	0.0134
0.12	0.0000	0.0000	0.0000	0.0057	0.0079	0.0100	0.0121
0.13	0.0000	0.0000	0.0000	0.0044	0.0066	0.0087	0.0108
0.14	0.0000	0.0000	0.0000	0.0000	0.0052	0.0074	0.0095
0.15	0.0000	0.0000	0.0000	0.0000	0.0000	0.0061	0.0082

Other parameters: $a = 0.1$, $\sigma = 0.01$, $\delta = 0.3$

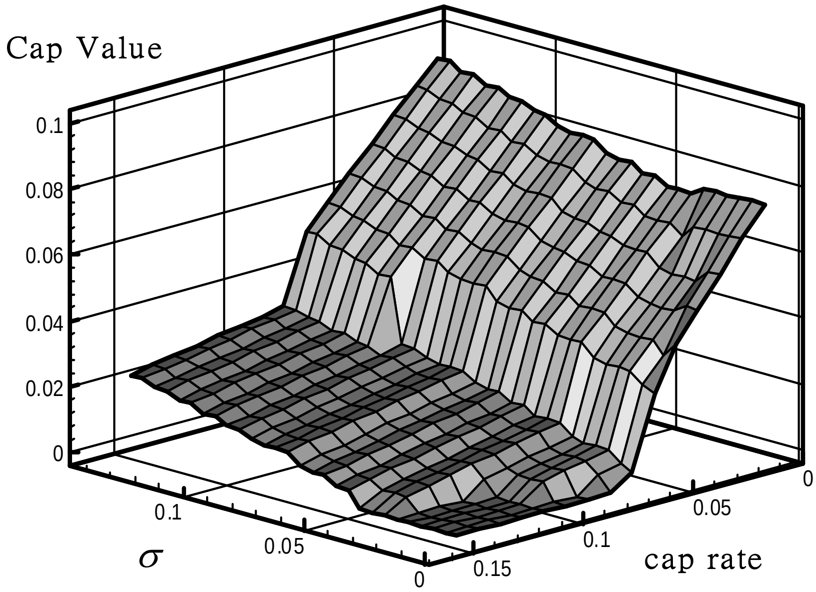

Figure 11 Cap on floating coupon bond

4. Conclusion

When counter party credit risk exists, default factor need to be considered in pricing process. This article uses two Markov processes, Hull-White model and Jarrow-Turnbull model, to build a combining risky interest rate tree for pricing risky interest rate derivatives. Meanwhile, the bankruptcy process is specified exogenously then the data is observable in the trading market. Upon determining the estimations of mean reversion and volatility of interest rate and acquiring default probabilities, the risky term structure could be accomplished. The risky tree, as a discrete combining lattice, not only fits the original term structure but also embraces the credit information of underlying assets. Upon satisfying both of advantages of HW model and JT model, it provides a useful method for pricing credit sensitive bonds and spread-adjusted contracts, such as interest rate options or swaps. The risky term structure could be widely extended to multi-factor interest derivatives, for example, mortgage or convertible bond.

In our simulation, we can see that mean reversion, interest rate volatility, and strike price (or cap rate) are important factors of valuing interest derivatives. Mean reversion and volatility determine the scope of interest term structure, on the other hand, strike price determine cash flow upon comparing floating interest rate level or the risky bond value. Notice that the change of recovery rate doesn't affect option value, neither put nor cap, since recovery rate is a tradeoff with default probability. Under similar credit risk, higher recovery rate just brings up higher default rate.

Since spot rate is the only endogenous factor of the tree, the lattice model may bring up some drawbacks of negative branching probability and negative martingale probability of default, especially in condition of excessive volatility of interest rate and irregular price movement of zero bond and corporate bond. For example, if the value of risky corporate bond is greater than government bond, the negative default probability may occur. Nevertheless, it doesn't affect the efficiency of pricing work.

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信用風險下之利率期限期構: 公司債賣權與利率上限之應用

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摘要

近來由於金融交易違約風險逐漸提高，使得在作資產價值衡量時，交易對手信用風險衡量格外受到重視。這一篇文章是主要以 Hull-White 模型和 Jarrow-Turnbull 模型為基礎，來建構一個風險性利率期限結構。此風險性利率樹模型不但反應出目前市場利率期限結構，並且將特定風險性資產的信用狀況考慮到模型中，用以合理地衡量金融商品價格。而模型的優點是可以透過簡單間斷計算方法下，求得未來各期的即期利率，而違約過程的參數估計也可直接從市場上觀察到的資料算出。另外，此模型可以廣泛地用在利率相關商品上，例如當我們在作具有信用風險敏感度的債券評價和利率價差計算時，都能透過風險性利率樹狀圖來求得。本篇重點除了在說明風險性利率期限結構的建構外，也將模型應用在兩個標地物資產是具有違約風險的利率衍生性金融商品，即公司債賣權和利率上限的評價。透過數值分析的模擬，來了解不同參數投入下對商品本身價值的影響。

關鍵字：利率期限結構、信用風險、三項樹、利率上限、賣權