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Research

Statistical Inference About the Shape Parameter of the New Two-parameter Bathtub-shaped Lifetime Distribution

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This paper proposes a simple and exact method for conducting a statistical test about the shape parameter of the new two-parameter lifetime distribution with a bathtub-shaped or increasing failure rate function, as well as an exact confidence interval for the same parameter. The necessary critical values of the test are given. The method provided in this paper can be used for type II right censored data. Moreover, Monte Carlo simulation and an example are used to compare this new method to the existing approach of Chen (Statistics and Probability Letters 2000; 49:155–161). Copyright © 2004 John Wiley & Sons, Ltd.

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1. INTRODUCTION

The class of lifetime distributions that have bathtub-shaped failure rate functions is very important because the lifetimes of electro-mechanical, electronic and mechanical products are often modeled with this characteristic. In survival analysis, the lifetime of humans exhibits this pattern. Thus, in recent years, some probability distributions have been proposed to fit real-life data with bathtub-shaped failure rates, such as Smith and Bain[1], Gaver and Acar[2], Hjorth[3], Leemis[4], Rajarshi and Rajarshi[5], Mudholkar and Srivastava[6], Mi[7] and Chen[8,9].

In this paper, we discuss the new two-parameter lifetime distribution with bathtub-shaped or increasing failure rate (IFR) function as proposed by Chen[9]. The cumulative distribution function (c.d.f.) of this distribution is

$$F(x) = 1 - \mathrm{e}^{\lambda(1-\mathrm{e}^{x^\beta})}, \quad x > 0 \tag{1}$$

where $\lambda > 0$ is the parameter but it does not affect the shape of the failure rate function, $h(x)$, as in Equation (2) and $\beta > 0$ is the shape parameter. The corresponding failure rate function of this distribution is

$$h(x) = \lambda\beta x^{\beta-1}\,\mathrm{e}^{x^\beta}, \quad x > 0 \tag{2}$$

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$$h'(x) = \lambda\beta x^{\beta-2}\,\mathrm{e}^{x^\beta}(\beta - 1 + \beta x^\beta), \quad x > 0 \tag{3}$$

$h(x)$ is bathtub shaped when $\beta < 1$ it achieves a minimum at $x = [(1-\beta)/\beta]^{1/\beta}$. The distribution has an IFR function when $\beta \geq 1$. Thus, the shape parameter β plays an important role in the distribution. Hence, the present paper proposes a simple exact statistical test with respect to the shape parameter β. An exact confidence interval for the shape parameter β is also discussed. The method can be used when a type II right-censored sample is available, i.e. the first k-order statistics of a sample of size n are given. Monte Carlo simulation was used to obtain the critical values necessary for conducting statistical tests or constructing confidence intervals for the shape parameter β. The simulation was based on 600 000 computer-generated pseudo-samples. The estimated critical values can be calculated using Compaq Visual Fortran version 6.5 and IMSL[10]. The Monte Carlo simulation results show that the average lengths of the confidence intervals for the shape parameter constructed by the method provided in the present paper are shorter that those constructed by the method in Chen[9]. The new proposed test is also uniformly more powerful than the test proposed by Chen[9]. Finally, we give an example that illustrates the procedure for conducting statistical tests or constructing confidence intervals for the shape parameter β.

2. MAIN RESULT

Let $X_{(1)}, \ldots, X_{(k)}$ be the first k-order statistics of a sample of size n from a population distribution with a c.d.f. as in Equation (1). It can be shown that

$$\lambda(\mathrm{e}^{X_{(1)}^\beta} - 1), \ldots, \lambda(\mathrm{e}^{X_{(k)}^\beta} - 1)$$

are the first k-order statistics of a sample of size n from a standard exponential distribution. Define

$$W(\beta; n, k) = \left\{\frac{1}{n}\left[\sum_{i=1}^{k-1}(\mathrm{e}^{X_{(i)}^\beta} - 1) + (n-k+1)(\mathrm{e}^{X_{(k)}^\beta} - 1)\right]\right\}\left\{\left[\prod_{i=1}^{k-1}(\mathrm{e}^{X_{(i)}^\beta} - 1)\cdot(\mathrm{e}^{X_{(k)}^\beta} - 1)^{n-k+1}\right]^{1/n}\right\}^{-1} \tag{4}$$

It is easy to show that the distribution of $W(\beta; n, k)$ does not depend on (β, λ) and therefore it provides a pivotal quantity for β. Let $W_\alpha(n, k)$ be the upper α percentile of the distribution of the pivotal quantity $W(\beta; n, k)$. Then

$$P(W_{1-\alpha/2}(n, k) < W(\beta; n, k) < W_{\alpha/2}(n, k)) = 1 - \alpha \tag{5}$$

for any $0 < \alpha < 1$. To derive a statistical test for the shape parameter β, the following lemma is necessary.

Lemma 1. *Suppose that $a_1, \ldots, a_k$ are positive, real numbers and that these numbers are not all the same. Then the function*

$$\phi(\beta; n, k) = \left\{\frac{1}{n}\left[\sum_{i=1}^{k-1}(\mathrm{e}^{a_i^\beta} - 1) + (n-k+1)(\mathrm{e}^{a_k^\beta} - 1)\right]\right\}\left\{\left[\prod_{i=1}^{k-1}(\mathrm{e}^{a_i^\beta} - 1)(\mathrm{e}^{a_k^\beta} - 1)^{n-k+1}\right]^{1/n}\right\}^{-1}$$

is strictly increasing in $\beta > 0$.

Proof. The function $\phi(\beta; n, k)$ is strictly increasing in β if and only if

$$\log\phi(\beta; n, k) = \log\left\{\frac{1}{n}\left[\sum_{i=1}^{k-1}(\mathrm{e}^{a_i^\beta} - 1) + (n-k+1)(\mathrm{e}^{a_k^\beta} - 1)\right]\right\}$$
$$- \frac{1}{n}\left[\sum_{i=1}^{k-1}\log(\mathrm{e}^{a_i^\beta} - 1) + (n-k+1)\log(\mathrm{e}^{a_k^\beta} - 1)\right]$$

is strictly increasing in β. Hence, it is sufficient to show that the derivative of $\log \phi(\beta; n, k)$,

$$\begin{aligned}\frac{\partial \log \phi(\beta; n, k)}{\partial \beta} &= \left\{\frac{1}{n}\left[\sum_{i=1}^{k-1} \mathrm{e}^{a_i^\beta} a_i^\beta \log a_i + (n-k+1)\,\mathrm{e}^{a_k^\beta} a_k^\beta \log a_k\right]\right\} \\ &\quad \times \left\{\frac{1}{n}\left[\sum_{i=1}^{k-1}(\mathrm{e}^{a_i^\beta} - 1) + (n-k+1)(\mathrm{e}^{a_k^\beta} - 1)\right]\right\}^{-1} \\ &\quad - \frac{1}{n}\left[\sum_{i=1}^{k-1} \frac{\mathrm{e}^{a_i^\beta} a_i^\beta \log a_i}{\mathrm{e}^{a_i^\beta} - 1} + (n-k+1)\frac{\mathrm{e}^{a_k^\beta} a_k^\beta \log a_k}{\mathrm{e}^{a_k^\beta} - 1}\right] \qquad (6)\end{aligned}$$

is positive.

Let Y be a random variable with the probability density function

$$f_Y(y) = \begin{cases} \dfrac{1}{n}, & y = a_1, a_2, \ldots, a_{k-1} \\ \dfrac{n-k+1}{n}, & y = a_k \end{cases}$$

In addition, we also define two functions of y

$$g(y) = \mathrm{e}^{y^\beta} - 1$$

and

$$h(y) = \frac{\mathrm{e}^{y^\beta} y^\beta \log y}{\mathrm{e}^{y^\beta} - 1}$$

Therefore, Equation (6) can be rewritten as

$$\frac{\partial \log \phi(\beta; n, k)}{\partial \beta} = \frac{E[g(Y)h(Y)]}{E(g(Y))} - E(h(Y))$$

By using functions $g(y)$ and $h(y)$ that are strictly increasing in $y > 0$ and a covariance inequality (see Casella and Berger[11] (p. 184)), we obtain $\partial \log \phi(\beta; n, k)/\partial\beta > 0$. The proof follows. □

Remark. If $0 \le a_1 \le a_2 \le \cdots \le a_k$, then the result of Lemma 1 is also true.

Now suppose that $X_{(1)}, \ldots, X_{(k)}$ are the first k-order statistics of a sample of size n from a population distribution with a c.d.f. as defined in Equation (1). Then the decision rule for the statistical test

$$H_0 : \beta = \beta_0 \quad \text{versus} \quad H_a : \beta \neq \beta_0$$

is to reject H_0 if

$$W(\beta_0; n, k) > W_{\alpha/2}(n, k) \quad \text{or} \quad W(\beta_0; n, k) < W_{1-\alpha/2}(n, k).$$

The upper percentiles of $W(\beta; n, k)$ were estimated by Monte Carlo simulation. The simulation used 600 000 replicates for each combination of n and k with $3 \le k \le n$, $n = 5$; $n/2 \le k \le n$, $n = 10, 20, 40$; $10 \le k \le n$, $n = 15, 30$ and various values of α. Some of these upper percentiles of $W(\beta; n, k)$ are given in Table I.

Lemma 2. *For $t > 1$, the equation*

$$\phi(\beta; n, k) = t$$

has a unique solution for $\beta > 0$ where $\phi(\beta; n, k) = t$ is defined in Lemma 1.

Proof. As $a_1, \ldots, a_k$ are not all the same, then $\phi(\beta; n, k) = 1$ if and only if $\beta = 0$. Hence, this lemma is a direct corollary of Lemma 1. □

Table I. Upper percentiles of $W(\beta; n, k)$

		α									
n	k	0.995	0.990	0.975	0.950	0.900	0.100	0.050	0.025	0.010	0.005
5	3	1.001	1.002	1.005	1.010	1.021	1.677	1.962	2.293	2.812	3.278
	4	1.008	1.012	1.023	1.039	1.066	2.042	2.439	2.899	3.618	4.261
	5	1.026	1.037	1.060	1.090	1.137	2.495	3.032	3.656	4.643	5.540
10	5	1.009	1.014	1.023	1.035	1.054	1.530	1.683	1.845	2.077	2.265
	6	1.020	1.027	1.042	1.059	1.086	1.662	1.841	2.031	2.300	2.522
	7	1.035	1.046	1.066	1.089	1.124	1.806	2.013	2.233	2.546	2.802
	8	1.054	1.069	1.096	1.125	1.168	1.963	2.202	2.455	2.816	3.111
	9	1.078	1.097	1.131	1.167	1.219	2.140	2.414	2.705	3.118	3.454
	10	1.109	1.132	1.174	1.218	1.280	2.346	2.663	2.998	3.475	3.866
15	10	1.061	1.074	1.098	1.123	1.158	1.712	1.861	2.011	2.217	2.381
	11	1.078	1.094	1.122	1.151	1.191	1.804	1.966	2.131	2.356	2.534
	12	1.098	1.116	1.148	1.181	1.226	1.901	2.079	2.260	2.506	2.703
	13	1.120	1.141	1.177	1.214	1.265	2.007	2.201	2.399	2.668	2.882
	14	1.145	1.169	1.210	1.252	1.308	2.124	2.337	2.553	2.849	3.084
	15	1.174	1.201	1.248	1.294	1.357	2.254	2.489	2.727	3.052	3.309
20	10	1.042	1.052	1.069	1.087	1.111	1.486	1.581	1.675	1.802	1.901
	11	1.054	1.065	1.085	1.105	1.133	1.541	1.643	1.746	1.882	1.987
	12	1.067	1.080	1.102	1.125	1.156	1.599	1.710	1.819	1.966	2.079
	13	1.081	1.096	1.121	1.146	1.181	1.660	1.778	1.895	2.053	2.176
	14	1.097	1.113	1.141	1.169	1.207	1.723	1.850	1.976	2.144	2.274
	15	1.114	1.132	1.163	1.193	1.234	1.789	1.925	2.060	2.240	2.379
	16	1.132	1.152	1.186	1.219	1.264	1.860	2.005	2.149	2.340	2.489
	17	1.151	1.174	1.211	1.247	1.296	1.934	2.088	2.243	2.449	2.606
	18	1.173	1.197	1.237	1.277	1.329	2.013	2.179	2.343	2.563	2.733
	19	1.197	1.223	1.266	1.309	1.366	2.099	2.276	2.452	2.689	2.874
	20	1.223	1.252	1.299	1.346	1.407	2.195	2.385	2.574	2.826	3.021
30	10	1.027	1.033	1.043	1.055	1.070	1.296	1.351	1.404	1.474	1.526
	11	1.034	1.041	1.053	1.066	1.084	1.328	1.385	1.442	1.516	1.572
	12	1.042	1.050	1.064	1.078	1.097	1.359	1.420	1.480	1.558	1.617
	13	1.050	1.059	1.075	1.091	1.112	1.391	1.456	1.519	1.602	1.664
	14	1.059	1.069	1.086	1.104	1.127	1.424	1.492	1.559	1.646	1.712
	15	1.069	1.080	1.099	1.117	1.142	1.457	1.530	1.600	1.691	1.760
	16	1.079	1.091	1.112	1.132	1.158	1.492	1.568	1.641	1.737	1.809
	17	1.090	1.103	1.125	1.147	1.175	1.527	1.607	1.684	1.785	1.860
	18	1.101	1.115	1.139	1.162	1.192	1.563	1.647	1.728	1.834	1.913
	19	1.113	1.128	1.154	1.178	1.210	1.601	1.688	1.773	1.884	1.968
	20	1.126	1.142	1.169	1.195	1.229	1.639	1.730	1.819	1.934	2.022
	21	1.139	1.156	1.185	1.212	1.249	1.679	1.775	1.867	1.989	2.080
	22	1.153	1.171	1.201	1.231	1.269	1.720	1.820	1.917	2.044	2.138
	23	1.167	1.187	1.219	1.250	1.290	1.762	1.867	1.968	2.100	2.200
	24	1.182	1.203	1.237	1.269	1.312	1.807	1.916	2.021	2.160	2.264
	25	1.198	1.220	1.255	1.290	1.335	1.852	1.967	2.078	2.222	2.330
	26	1.214	1.237	1.275	1.311	1.358	1.900	2.019	2.135	2.286	2.399
	27	1.232	1.256	1.296	1.334	1.383	1.950	2.076	2.197	2.355	2.474
	28	1.250	1.276	1.318	1.358	1.410	2.003	2.133	2.260	2.424	2.547
	29	1.271	1.298	1.342	1.384	1.438	2.060	2.196	2.328	2.501	2.632
	30	1.292	1.321	1.367	1.412	1.469	2.121	2.264	2.403	2.583	2.720

Lemma 2 can also be used to construct an exact $1-\alpha$ confidence interval of the shape parameter β. The confidence interval can be expressed as (β_L, β_U), where β_L is the solution of β for the equation

$$W(\beta; n, k) = W_{1-\alpha/2}(n, k)$$

Table I. (*Continued*)

		α									
n	k	0.995	0.990	0.975	0.950	0.900	0.100	0.050	0.025	0.010	0.005
40	20	1.089	1.100	1.119	1.138	1.162	1.439	1.498	1.555	1.629	1.683
	21	1.098	1.110	1.130	1.149	1.175	1.463	1.525	1.584	1.660	1.717
	22	1.107	1.120	1.141	1.161	1.188	1.489	1.553	1.614	1.693	1.751
	23	1.116	1.130	1.152	1.174	1.201	1.514	1.581	1.644	1.726	1.787
	24	1.126	1.141	1.164	1.186	1.215	1.540	1.609	1.675	1.760	1.823
	25	1.136	1.151	1.176	1.199	1.229	1.567	1.638	1.706	1.794	1.859
	26	1.147	1.162	1.188	1.213	1.244	1.594	1.668	1.739	1.830	1.898
	27	1.157	1.174	1.201	1.226	1.259	1.622	1.699	1.772	1.866	1.935
	28	1.168	1.186	1.214	1.241	1.275	1.651	1.730	1.805	1.903	1.975
	29	1.180	1.198	1.227	1.255	1.291	1.680	1.762	1.840	1.940	2.014
	30	1.192	1.211	1.241	1.270	1.307	1.711	1.795	1.875	1.979	2.055
	31	1.204	1.224	1.256	1.286	1.324	1.742	1.829	1.912	2.019	2.098
	32	1.217	1.237	1.271	1.302	1.341	1.773	1.864	1.950	2.059	2.142
	33	1.230	1.251	1.286	1.318	1.359	1.807	1.900	1.989	2.103	2.187
	34	1.244	1.266	1.301	1.335	1.378	1.840	1.936	2.029	2.147	2.235
	35	1.258	1.281	1.318	1.353	1.397	1.875	1.975	2.070	2.192	2.281
	36	1.273	1.297	1.335	1.371	1.417	1.912	2.015	2.113	2.238	2.331
	37	1.288	1.313	1.353	1.390	1.438	1.949	2.056	2.157	2.287	2.383
	38	1.304	1.330	1.371	1.410	1.460	1.989	2.099	2.204	2.338	2.438
	39	1.321	1.348	1.391	1.432	1.483	2.031	2.145	2.254	2.392	2.496
	40	1.339	1.367	1.412	1.454	1.508	2.077	2.194	2.307	2.451	2.559

and β_U is the solution of β for the equation

$$W(\beta; n, k) = W_{\alpha/2}(n, k)$$

The unique solution of β for the equation $W(\beta; n, k) = t\,(t > 1)$ can be obtained by using Compaq Visual Fortran version 6.5 and the subroutine DZREAL of IMSL[10].

3. COMPARISON AND EXAMPLE

Chen[9] proposed an exact confidence interval for the shape parameter β of a population distribution with a c.d.f. as in Equation (1)

$$(\varphi(X_{(1)}, \ldots, X_{(k)}, F_{1-\alpha/2}(2k-2, 2)), \quad \varphi(X_{(1)}, \ldots, X_{(k)}, F_{\alpha/2}(2k-2, 2)))$$

where $\varphi(X_{(1)}, \ldots, X_{(k)}, t)$ is the solution for β of the equation

$$\frac{\sum_{i=2}^{k-1}(e^{X_{(i)}^{\beta}} - 1) + (n-k+1)(e^{X_{(k)}^{\beta}} - 1) + (1-n)(e^{X_{(1)}^{\beta}} - 1)}{n(k-1)(e^{X_{(1)}^{\beta}} - 1)} = t$$

To compare the method provided in the present paper with the one given in Chen[9], 10 000 random samples from a population distribution with a c.d.f. as defined in Equation (1) with parameters $(\lambda, \beta) = (1, 0.4), (1, 1)$ and $(1, 1.2)$ were generated for several combinations of n and k. Under the parameters $(\lambda, \beta) = (1, 0.4), (1, 1)$ and $(1, 1.2)$, the average lengths of 90% confidence intervals constructed by these two methods were listed in Table II. We found that our proposed method yielded shorter average lengths than the method of Chen[9] for all given combinations of n and k. Tables III and IV show power comparisons between these two methods for the hypothesis $H_0 : \beta = \beta_0$ versus $H_a : \beta \neq \beta_0$ at the level of significance of $\alpha = 0.1$ and $\lambda = 1$ for $\beta_0 = 0.4$

Table II. Average length of 90% confidence interval of β under $\lambda = 1$

n	k	β	Method	Average length	n	K	β	Method	Average length
10	5	0.4	Chen	1.01	30	20	0.4	Chen	0.53
			New	0.86				New	0.33
		1	Chen	2.55			1	Chen	1.33
			New	2.24				New	0.83
		1.2	Chen	3.09			1.2	Chen	1.60
			New	2.78				New	1.00
10	10	0.4	Chen	0.61	30	30	0.4	Chen	0.42
			New	0.42				New	0.22
		1	Chen	1.52			1	Chen	1.05
			New	1.05				New	0.55
		1.2	Chen	1.82			1.2	Chen	1.27
			New	1.27				New	0.66
20	10	0.4	Chen	0.70	40	20	0.4	Chen	0.52
			New	0.52				New	0.34
		1	Chen	1.75			1	Chen	1.33
			New	1.31				New	0.86
		1.2	Chen	2.12			1.2	Chen	1.61
			New	1.56				New	1.03
20	20	0.4	Chen	0.47	40	30	0.4	Chen	0.46
			New	0.27				New	0.26
		1	Chen	1.18			1	Chen	1.16
			New	0.69				New	0.65
		1.2	Chen	1.43			1.2	Chen	1.40
			New	0.83				New	0.78
30	10	0.4	Chen	0.71	40	40	0.4	Chen	0.39
			New	0.53				New	0.19
		1	Chen	1.74			1	Chen	0.98
			New	1.32				New	0.47
		1.2	Chen	2.10			1.2	Chen	1.18
			New	1.57				New	0.56

and 1, respectively. We found that the new proposed test is uniformly more powerful than the test of Chen[9] for all given combinations of n and k. Hence, it can be seen from these tables that the new method out performs that of Chen.

Example. The following are the first $k = 11$ observations of a computer-generated sample of size $n = 15$ from a population distribution defined by Equation (1) with parameters $\lambda = 0.02$ and $\beta = 0.5$ (also see Chen[9]):

0.29, 1.44, 8.38, 8.66, 10.20, 11.04, 13.44, 14.37, 17.05, 17.13 and 18.35

It was found by Chen[9] that (0.19, 0.62) is a 95% confidence interval with interval length 0.43 for the shape parameter β. Now we want to construct a 95% confidence interval for the same parameter β using the method presented here. From Table I, the values of $W_{0.975}(15, 11)$ and $W_{0.025}(15, 11)$ are 1.122 and 2.131, respectively. The numerical solution for β of the equation

$$\left\{\frac{1}{15}\left[\sum_{i=1}^{10}(e^{X_{(i)}^{\beta}}-1)+5(e^{X_{(11)}^{\beta}}-1)\right]\right\}\left\{\left[\prod_{i=1}^{10}(e^{X_{(i)}^{\beta}}-1)(e^{X_{(11)}^{\beta}}-1)^5\right]^{1/15}\right\}^{-1}=1.122$$

is 0.27, and the numerical solution for β of the equation

$$\left\{\frac{1}{15}\left[\sum_{i=1}^{10}(e^{X_{(i)}^{\beta}}-1)+5(e^{X_{(11)}^{\beta}}-1)\right]\right\}\left\{\left[\prod_{i=1}^{10}(e^{X_{(i)}^{\beta}}-1)(e^{X_{(11)}^{\beta}}-1)^5\right]^{1/15}\right\}^{-1}=2.131$$

Table III. Power comparisons for $H_0 : \beta = 0.4$ versus $H_a : \beta \neq 0.4$ at $\alpha = 0.1$ and $\lambda = 1$

			β									
n	k	Method	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
10	5	Chen	0.79	0.37	0.16	0.10	0.10	0.14	0.18	0.24	0.30	0.34
		New	0.96	0.44	0.18	0.10	0.10	0.14	0.19	0.25	0.32	0.37
10	10	Chen	0.97	0.55	0.19	0.10	0.14	0.24	0.36	0.46	0.56	0.66
		New	1.00	0.79	0.31	0.10	0.16	0.32	0.50	0.66	0.79	0.87
20	10	Chen	0.93	0.48	0.18	0.10	0.12	0.21	0.30	0.41	0.50	0.59
		New	0.99	0.67	0.25	0.10	0.13	0.24	0.37	0.51	0.65	0.75
20	15	Chen	0.98	0.56	0.20	0.10	0.16	0.27	0.39	0.53	0.65	0.73
		New	1.00	0.83	0.32	0.10	0.18	0.37	0.59	0.77	0.88	0.94
20	20	Chen	0.99	0.69	0.22	0.10	0.18	0.32	0.50	0.63	0.75	0.82
		New	1.00	0.96	0.46	0.10	0.25	0.57	0.82	0.94	0.98	1.00
30	10	Chen	0.97	0.49	0.18	0.10	0.13	0.19	0.30	0.41	0.49	0.57
		New	0.97	0.66	0.25	0.10	0.13	0.24	0.37	0.50	0.61	0.72
30	20	Chen	0.99	0.61	0.21	0.10	0.16	0.30	0.46	0.60	0.72	0.79
		New	1.00	0.90	0.36	0.10	0.21	0.45	0.70	0.86	0.95	0.98
30	30	Chen	1.00	0.76	0.22	0.10	0.21	0.39	0.58	0.74	0.82	0.89
		New	1.00	0.99	0.59	0.10	0.35	0.76	0.95	0.99	1.00	1.00
40	20	Chen	0.99	0.61	0.20	0.10	0.15	0.30	0.44	0.59	0.71	0.80
		New	1.00	0.88	0.34	0.10	0.20	0.42	0.66	0.84	0.93	0.98
40	30	Chen	1.00	0.69	0.22	0.11	0.19	0.37	0.56	0.70	0.81	0.88
		New	1.00	0.97	0.48	0.10	0.28	0.64	0.89	0.97	1.00	1.00
40	40	Chen	1.00	0.81	0.24	0.09	0.22	0.44	0.64	0.78	0.87	0.93
		New	1.00	1.00	0.69	0.10	0.43	0.87	0.99	1.00	1.00	1.00

Table IV. Power comparisons for $H_0 : \beta = 1$ versus $H_a : \beta \neq 1$ at $\alpha = 0.1$ and $\lambda = 1$

			β									
n	k	Method	0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2
10	5	Chen	0.89	0.52	0.25	0.14	0.10	0.10	0.12	0.15	0.19	0.23
		New	0.94	0.60	0.30	0.15	0.10	0.10	0.12	0.16	0.20	0.25
10	10	Chen	1.00	0.77	0.36	0.15	0.10	0.13	0.20	0.28	0.38	0.45
		New	1.00	0.92	0.59	0.23	0.10	0.14	0.24	0.39	0.53	0.65
20	10	Chen	0.98	0.68	0.33	0.14	0.10	0.12	0.17	0.24	0.32	0.40
		New	1.00	0.84	0.47	0.19	0.10	0.13	0.19	0.29	0.40	0.51
20	15	Chen	1.00	0.77	0.37	0.16	0.11	0.13	0.21	0.32	0.43	0.54
		New	1.00	0.95	0.62	0.24	0.10	0.15	0.28	0.47	0.62	0.76
20	20	Chen	1.00	0.89	0.44	0.17	0.11	0.16	0.27	0.40	0.53	0.64
		New	1.00	1.00	0.84	0.35	0.10	0.20	0.45	0.70	0.86	0.94
30	10	Chen	0.98	0.68	0.32	0.15	0.09	0.12	0.17	0.24	0.31	0.40
		New	1.00	0.84	0.46	0.19	0.10	0.12	0.19	0.29	0.38	0.49
30	20	Chen	1.00	0.83	0.40	0.16	0.10	0.14	0.23	0.37	0.49	0.61
		New	1.00	0.98	0.71	0.27	0.10	0.17	0.35	0.56	0.74	0.86
30	30	Chen	1.00	0.95	0.51	0.18	0.10	0.17	0.32	0.46	0.61	0.73
		New	1.00	1.00	0.94	0.43	0.10	0.26	0.61	0.85	0.96	0.99
40	20	Chen	1.00	0.83	0.40	0.16	0.10	0.14	0.24	0.35	0.47	0.59
		New	1.00	0.98	0.69	0.26	0.10	0.16	0.33	0.53	0.71	0.84
40	30	Chen	1.00	0.90	0.46	0.16	0.10	0.16	0.29	0.44	0.58	0.70
		New	1.00	1.00	0.86	0.35	0.10	0.22	0.50	0.76	0.91	0.97
40	40	Chen	1.00	0.97	0.54	0.19	0.10	0.18	0.35	0.52	0.68	0.77
		New	1.00	1.00	0.98	0.52	0.10	0.32	0.72	0.94	0.99	1.00

is 0.60 using Compaq Visual Fortran version 6.5 and the subroutine DZREAL of IMSL[10] (see Appendix A). Thus, a 95% confidence interval for the shape parameter β is (0.27, 0.60) with interval length 0.33. This provides evidence that the data set is probably better fit by a bathtub-shaped distribution than an IFR distribution since 1 is not in this interval. We found that our proposed method gives a shorter interval length than the method of Chen[9].

To test the hypothesis $H_0 : \beta = 0.5$ versus $H_a : \beta \neq 0.5$ at the level of significance of $\alpha = 0.1$, note that the value of the test statistic is

$$W(0.5;\, 15,\, 11) = \left\{ \frac{1}{15} \left[\sum_{i=1}^{10} (e^{X_{(i)}^{0.5}} - 1) + 5(e^{X_{(11)}^{0.5}} - 1) \right] \right\} \left\{ \left[\prod_{i=1}^{10} (e^{X_{(i)}^{0.5}} - 1)(e^{X_{(11)}^{0.5}} - 1)^5 \right]^{1/15} \right\}^{-1} = 1.581$$

It can be found from Table I that

$$W_{0.95}(15,\, 11) = 1.151 \quad \text{and} \quad W_{0.05}(15,\, 11) = 1.966$$

Thus, we fail to reject the null hypothesis at the level of significance of $\alpha = 0.1$. This conclusion is the same as Chen[9].

4. CONCLUSIONS

In this article, we propose a simple exact statistical test for the shape parameter β of a population distribution with a c.d.f. defined by Equation (1) based on the pivotal quantity of Equation (4). The new test is quite attractive because it is computationally simple and seems to have a reasonable level of performance for all given combinations of n and k checked here. One can also easily extend the method to test the null hypothesis $H_0 : \beta = \beta_0$ by using the multiply type II censored sample where the multiply type II censored sample supposes that the first r, last s and middle l observations are censored and the only observations available are $X_{r+1} < \cdots < X_{r+k}$ and $X_{r+k+l+1} < \cdots < X_{n-s}$.

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APPENDIX A

As an example, we use the following program to find the lower bound β_L and upper bound β_U of the 95% confidence interval for the shape parameter β based on Compaq Visual Fortran version 6.5 and the subroutine DZREAL of IMSL[11]:

```
      PROGRAM SOLUTION
      USE IMSL
      REAL*8 EPS,ERRABS,ERRREL,ETA
      PARAMETER (NROOT=1)
      INTEGER INFO(NROOT)
      REAL*8 F,F1,YGUESS(NROOT),X(11),Y,Y1
      EXTERNAL F,F1
      COMMON X
      DATA YGUESS/0.5/
      EPS=1.0E-5
      ERRABS=1.0E-5
      ERRREL=1.0E-6
      ETA=1.0E-5
      ITMAX=500
      X(1)=0.29
      X(2)=1.44
      X(3)=8.38
      X(4)=8.66
      X(5)=10.20
      X(6)=11.04
      X(7)=13.44
      X(8)=14.37
      X(9)=17.05
      X(10)=17.13
      X(11)=18.35
c     *** find the lower bound βL and upper bound βU of 95% confidence interval for β  ***
      CALL DZREAL(F,ERRABS,ERRREL,EPS,ETA,NROOT,ITMAX,YGUESS,Y,INFO)
      CALL DZREAL(F1,ERRABS,ERRREL,EPS,ETA,NROOT,ITMAX,YGUESS,Y1,INFO)
      WRITE(6,5)Y,Y1
5     FORMAT(2F7.2)
      STOP
      END

      REAL*8 FUNCTION F(Y)
      REAL*8 X(11),Y,M1,M2,M3,M4
      COMMON X
      M1=0.D0
      M3=1.D0
      DO 1 I=1,10
1         M1=M1+(EXP(X(I)**Y)-1.)
          M2=(M1+5.0*(EXP(X(11)**Y)-1.))/15.
      DO 2 I=1,10
2         M3=M3*((EXP(X(I)**Y)-1.)**(1./15.))
          M4=M3*(((EXP(X(11)**Y)-1.)**5.)**(1./15.))
      F=M2/M4-1.122
```



```
      RETURN
      END

      REAL*8 FUNCTION F1(Y1)
      REAL*8 X(11),Y1,M1,M2,M3,M4
      COMMON X
      M1=0.
      M3=1.
      DO 3 I=1,10
3         M1=M1+(EXP(X(I)**Y1)-1.)
          M2=(M1+5.0*(EXP(X(11)**Y1)-1.))/15.
      DO 4 I=1,10
4         M3=M3*((EXP(X(I)**Y1)-1.)**(1./15.))
          M4=M3*(((EXP(X(11)**Y1)-1.)**5.)**(1./15.))
      F1=M2/M4-2.131
      RETURN
      END
```

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