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Music walk, fractal geometry in music

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Abstract

In this study, sequences of musical notes from various pieces of music are converted into one-variable random walks (here termed 'music walks'). Quantitative measurements of the properties of each musical composition are then performed by applying Hurst exponent and Fourier spectral analyses on these music-walk sequences. Our results show that music shares the similar fractal properties of a fractional Brownian motion (fBm). That is, music displays an anti-persistent trend in its tone changes (melody) over decades of musical notes; and music sequence exhibits generally the $1/f^{\beta}$-type spectrum (fractal property), with apparently two different β values in two different temporal scales.

Keywords: Music walk; Fractal; Fractional Brownian motion (fBm); Long-range correlation; Hurst exponent; Power spectrum

1. Introduction

There are almost as many styles of music as there are composers, as each composer imparts their own creative preferences and ideas when working on a composition. However, beyond this variety and individuality, are there rules or an underlying structure that essentially differentiates a musical work from meaningless collections of notes? This hypothesis can be investigated through quantitative analysis of the fractal properties of musical compositions. Although the mathematical theory of music is rather deep and complicated [1], quantitative study can nevertheless be performed by appealing to the notion of fractional Brownian motion (fBm) and Fourier spectral analysis. Interestingly, as disclosed in this study, the results show that music exhibits the ubiquitous property of long-term correlation (fractal) over decades of notes in two apparently different scaling ranges. This underlying structure may explain why music sounds generally pleasing, and shows how music imitates the harmony of nature.

2. Literature review

Music is conventionally defined as an ordered arrangement of sounds of different acoustic frequencies (pitches, tones) in succession (melody), of sounds in combination (harmony), and of sounds spaced in

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temporal succession (rhythm) [2]. An individual pitch (or tone) does not by itself produce music; it is the melody (successive changes in pitch) and rhythm (successive changes in tone duration) that are the two essential elements of music. The arithmetical relationships of notes in harmony was speculated upon at length by numerologists in the middle Ages, and it has been known for thousands of years that notes of a frequency ratio 2:1 produce the octave. Much of the theoretical and experimental work about music and tuning was performed in between 15th and 17th centuries. For example, in dividing an octave, one of the most popular methods considered desirable (though not essential) in order to play a musical composition harmonically in all keys is the equal temperament rule [1,2]. If f_0 denotes the pitch (frequency) of the base note (e.g. middle C), a note an octave higher than the base note has a pitch ratio $f/f_0 = 2$ with reference to the base note. In the equal temperament rule, an octave is divided into 12 geometrically equal intervals in pitch (i.e., the 7 white and 5 black keys on a piano) according to:

$$f_i/f_0 = 2^{i/12}. \tag{1}$$

Thus a semitone is represented by the pitch interval $i = 1$, a full tone by $i = 2$, a minor third by $i = 3$, a major third by $i = 4$, a fourth by $i = 5$, a diminished fifth by $i = 6$, a sonorous fifth by $i = 7$ and an octave by $i = 12$. In such a calibration, each note has a frequency ratio (with reference to the base note) of approximately a ratio of small integers. For instance, a fourth ($i = 5$) has a ratio of $2^{5/12} = 1.3348 \approx \frac{4}{3}$; a fifth ($i = 7$) has a ratio of $2^{7/12} = 1.4983 \approx \frac{3}{2}$. The exception is the diminished fifth ($i = 6$), which has a ratio $2^{6/12} = 1.4142 \approx 1000/707$, which is obviously not a ratio of small integers. This note interval has traditionally been considered dissonant and thus has rarely been used in classical compositions [2].

Recent analysis of musical structure has revealed evidence of a long-range scaling property similar to that found in natural landscapes, such as the profiles of mountains, coastlines, etc. [3]. By performing power spectral analysis upon the audio signals recorded from various selected pieces of different music styles (classical, rock, jazz and blues), Voss and Clarke [4,5] have shown that the power spectra of both the instantaneous loudness and the frequency fluctuations (roughly approximating to the melody) of music vary approximately as $1/f$, a typical characteristic of a scaling noise found in electronic components [6,7]. Such a tendency has also been confirmed by Schroeder [8] and Campbell [9], whose results imply the existence of a certain long-range correlation, fluctuating in quantity according to piece and style, over a major part of the music.

That music shows a $1/f$-spectrum is not greatly surprising, because $1/f^0$-dependence signal (random noise) sounds meaningless, while $1/f^2$-dependence signal (Brownian noise) would sound a little bit dull [10]. Therefore, $1/f$-noise shows a greater correlation in adjacent signal values than random noise and a weaker correlation when compared with Brownian noise [11]; and the property of $1/f$-dependence signal appears to serve as a compromise between too little and too much 'surprise', randomness or musicality in a musical composition. Music is a blend of randomness and orderliness. Consequently, $1/f$-signal is a good candidate for stochastic music composition [12–15].

Several researchers have made use of quantitative methods in order to study the properties of music. Hsu and Hsu [16,17] (see also Ref. [2]) analyzed the variations in pitch interval (the i defined in Eq. (1)) between successive notes in a series of music scores composed by Bach and Mozart, and were able to show that the incidence frequency (the frequency of appearance of each pitch interval) F approximately shows a power-law relationship, $F \propto i^{-D}$. The value of the exponent D for each different score varies between 1 and 3 and is not an integer; in other words, the incidence of change in acoustic frequency in a section of a music score exhibits fractal geometry. The authors also considered the use of this fractal property in an attempt to reduce the length of a composition while still being able to maintain its style.

In a recent study, Shi [18] applied two different types of correlation analyses, frequency dependent and frequency independent, to examine the music sequences converted from folk songs and piano pieces. His results indicated that music sequences have a long-range power-law behavior in both analyses, and that the fundamental principle of music is the obtaining of a balance between repetition and contrast. Further, Bigerelle and Iost [19] applied the "variance method" to study the fractal dimensions in 180 musical works of various styles. Based on their statistical results, they proposed that various music pieces could be categorized by fractal dimension. Madison [20] used a similar approach to study different musical scores, and found that the Hurst exponent plays an important role in examining the emotional expression of a musical performance.

The study by Manaris et al. [21] of a 220-piece corpus (including baroque, classical, romantic, 12-tone, jazz, rock, DNA strings, and random music) revealed that esthetically pleasing music might be describable under the Zipf–Mandelbrot law. Gündüz and Gündüz [22] studied the mathematical structures of six songs by treating them as complex systems. The key approach of their analysis was to calculate the fractal dimension of a scattering diagram constructed from the melody of the six songs.

In this paper, the sequence of musical notes is converted into a one-variable random walk (here termed the 'music walk'). The resulting graph appears to resemble a mountainous landscape in profile view, with jagged ridges of all lengths (scales), from minor bumps to enormous peaks. The fractal property of the 'music walk' is then explored by adopting the notion of fBm proposed by Mandelbrot and Van Ness [23], whose description is given in the following section.

3. Properties of fBm

One common method for extracting hidden structure information from a fluctuating time-series in signal processing is to calculate its power spectrum distribution. By applying the Fourier transform to a given continuous time series $x(t)$, the time-distribution signal is converted into a frequency-distribution one

$$x(f) \propto \int x(t) e^{-i2\pi ft}\, dt \tag{2}$$

and the power spectrum of the signal $x(t)$ is obtained as

$$S_x(f) \propto |x(f)|^2. \tag{3}$$

Another useful measurement of a time series is the autocorrelation function, $C_x(\tau)$, which is defined as

$$C_x(\tau) = \langle x(t) \cdot x(t+\tau) \rangle, \tag{4}$$

where $\langle \cdot \rangle$ denotes averaging the quantity over time t. The autocorrelation function is a measure of how well the data set correlates with itself after a time lag τ. The power spectrum and autocorrelation function of a time signal are not independent; they are related by the Wiener–Khintchine relations [24]

$$S_x(f) \propto \int C_x(\tau) \cos(2\pi f\tau)\, d\tau, \tag{5}$$

$$C_x(\tau) \propto \int S_x(f) \cos(2\pi f\tau)\, df. \tag{6}$$

The most commonly encountered fluctuating signal is the normally distributed random noise (also termed Gaussian white noise), $w(t)$. The autocorrelation function for such noise is a delta function $\delta(\tau)$; that is, successive signals in random noise are totally uncorrelated. The power spectrum distribution of random noise is flat ($S_w(f) \propto f^0$, a constant), indicating an equal composition of components of all frequencies. Integrating random noise with time then results in a new fluctuating time signal,

$$x_w(t) = \int w(t)\, dt, \tag{7}$$

where $x_w(t)$ can be regarded as the instantaneous location of a 'walker' who takes steps to the right or left along a one-dimensional line, with each step size determined by the successive signal values of a random noise $w(t)$. Such motion is termed a 'random walk' or 'Brownian motion', named after Robert Brown, a Scottish botanist. It is a well-known fact in statistics that the average distance traveled within a time period T for a random walk is given by the diffusion law

$$\overline{\Delta x_w(T)} = \langle |x_w(t+T) - x_w(t)|^2 \rangle^{1/2} \propto T^{1/2}. \tag{8}$$

The power spectrum of Brownian motion is therefore $S_{x_w}(f) \propto 1/f^2$. In contrast to random noise $w(t)$, successive values in a random walk signal $x_w(t)$ are strongly correlated.

Mandelbrot and Van Ness [23] generalized expression (8) into the form

$$\overline{\Delta x_H(T)} = \langle |x_H(t+T) - x_H(t)|^2 \rangle^{1/2} \propto T^H, \tag{9}$$

with $0 < H < 1$, H being the Hurst exponent. The corresponding motion (time signal) $x_H(t)$ is now generally termed fBm. For $H > 1/2$, the graph of fBm $x_H(t)$ is less rugged-looking, or smoother, than that of Brownian motion itself ($H = \frac{1}{2}$); and $x_H(t)$ tends to increase (decrease) in the future if it is increasing (decreasing) in the past, i.e., showing the property of 'persistence' in the increments of signal values Δx. For $H < \frac{1}{2}$, the graph of $x_H(t)$ is more rugged-looking and less smooth than that of Brownian motion; and $x_H(t)$ tends to decrease (increase) in the future if it is increasing (decreasing) in the past, i.e., showing a trend of 'anti-persistence' in the increments of signal values Δx. When $H = \frac{1}{2}$, the signal is restored to the conventional Brownian motion, $x_w(t)$, and increments of adjacent signal values (not the signal values themselves) in a Brownian motion are totally uncorrelated (random). The power spectrum of an fBm has the property

$$S_{x_H}(f) \propto 1/f^{\beta} \tag{10}$$

with the power β related to the Hurst exponent H by the relationship $\beta = 2H + 1$ [25,26], and $1 < \beta < 3$ for $0 < H < 1$.

According to the Wiener–Khintchine relations, the autocorrelation function $c_{x_H}(\tau)$ of the time series $x_H(t)$ having a $1/f^{\beta}$ power spectrum with β close to 1 also exhibits a power-law decaying behavior. The slowly decaying autocorrelation function implies that the time series are self-correlated over a rather long period of time; that is, the signal at any time is still related in a certain way to a signal appearing long before. The time signal is scale invariant; such a fluctuating signal is simply the time analogy of many naturally found self-similar geometries, which are now categorized under the name 'fractals'. The fine structure of fractal geometry, when magnified, looks like the structure as a whole, and in a fractal time signal, a short time section of the signal has the same statistical properties as the whole signal. Theoretically, the fractal dimension (box-counting dimension) D of the landscape traced by one-variable fBm and its Hurst exponent H is related by $D = 2 - H$ [25–29].

4. Music walk as an fBm

Choosing an arbitrary note (e.g. 'c', a note an octave below middle C) as a base note, all notes in a score can then subsequently be digitized and labeled by the index i, as defined in the equal temperament rule, for example, c with $i = 0$, $^{\#}c$ with $i = 1$, d with $i = 2, \ldots$ etc. Also, similar to the work of Shi [18], when considering the difference in tone duration of each note, the shortest time in the score is chosen as the step unit of the time-wise coordinate. That is, if the shortest time (beat) in a music score is $\frac{1}{16}$ note, then the index value representing the pitch of a $\frac{1}{4}$ note is repeated 4 times when constructing the time sequence from that music piece. The entire music score is thus converted into a reminiscence of a random walk (music walk), $x(n)$. The sequence $x(n)$ is simply the discrete version of the continuous time signal $x(t)$, with n being the number of notes (in multiplicity of the note with smallest time, as explained above) representing the elapsed time in a musical movement, and x being the value of the index i, which represents the pitch of the note. Therefore, both the melody (changes in pitch) and the rhythm (changes in tone duration) of a composition are taken into account in the digitized sequence $x(n)$. Note that Hsu and Hsu [16,17] did not include any effect of tone duration in their analysis. The typical $x(n)$ curves are shown in Fig. 1; these are the resulting curves from the conversion of the violin scores from *Gavotte* written by Gossec, *Le Cygne* written by Saint-Saëns and *Ave Maria* written by Bach and Gounod. As can be seen from Fig. 1, these plots resemble the fractal profiles of skyscrapers, or mountain ridges if x and n are properly scaled.

Characteristic of the fluctuating signal $x(n)$ is then studied by calculating the Hurst exponent H and the power β of the power spectrum distribution for various music-walk sequences converted from various music pieces: (I) *Gavotte* by Gossec; (II) *Le Cygne* by Saint-Saëns; (III) *Ave Maria* by Bach and Gounod; (IV) Beethoven, Op. 24, No. 5, *Sonata* for piano (IVa) and violin (IVb); (V) Mozart, Op. 70, No. 1, duo violins, violin 1 (Va) and violin 2 (Vb); (VI) Pachelbel's Canon and Gigue, 3 violins and continuo, violin 1 (VIa) and violin 2 (VIb); (VII) Bach, J. C., six duets, No. 1, duo violins, violin 1 (VIIa) and violin 2 (VIIb). In order to avoid the difficulty that might be encountered in selecting the proper 'key note' from a chord, the music scores

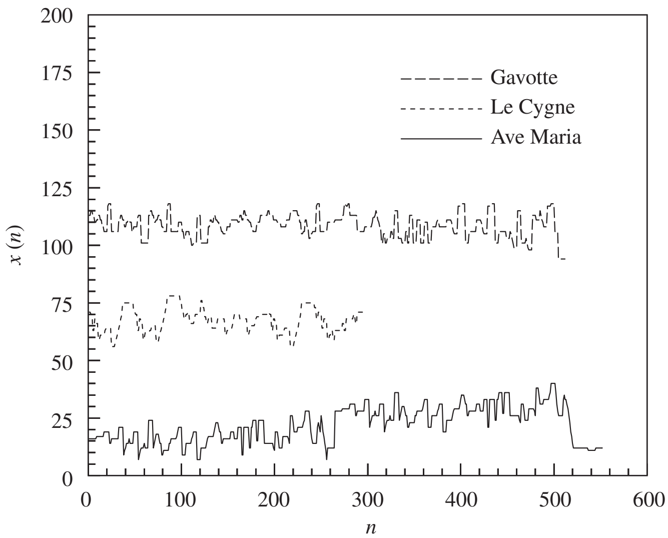

Fig. 1. Music-walk sequences obtained from the violin score of three different pieces of music, *Gavotte*, *Le Cygne* and *Ave Maria*. For clarity, the data values of *Le Cygne* and *Gavotte* have been shifted uniformly upward by 40 and 80 units, respectively.

Table 1
Values of the Hurst exponent H and exponent β of the power spectrum obtained from various music-walk sequences (short time-interval analysis)

Short time-interval analysis	I	II	III	IV(a)	IV(b)	V(a)	V(b)	VI(a)	VI(b)	VII(a)	VII(b)
H	0.35	0.38	0.22	0.38	0.34	0.22	0.22	0.23	0.26	0.32	0.30
β	1.80	1.67	1.43	1.74	1.83	1.50	1.64	1.57	1.55	1.54	1.51
$2H+1$	1.70	1.76	1.44	1.76	1.68	1.44	1.44	1.46	1.52	1.64	1.60
Deviation	5.88%	5.11%	0.69%	1.14%	8.93%	4.17%	13.89%	7.53%	1.97%	6.10%	5.63%

we choose to study in this paper are all violin scores (except IVa). The Hurst exponent H is obtained by taking logarithm of both sides of the discrete version of Eq. (9) to give

$$\log \sigma(\Delta x) = K + H \log |\Delta n|, \tag{11}$$

where $\sigma(\Delta x)$ is the standard deviation of the increments in pitch Δx corresponding to the interval size (number of notes) Δn, and K is a constant. Similarly, the β value is just the slope of the straight line obtained from least-square curve fitting of the power spectrum (in logarithmic scales) of the music-walk sequence $x(n)$. In this study, power spectra are obtained by applying discrete version of the Fourier transform (2) to music-walk sequences. Many commercialized numerical software packages provide subroutines for calculating power spectrum from given time sequence, such as MATLAB.

The results are summarized in Tables 1 and 2. Graphical representations of the results from the first three music pieces I, II and III are shown in Figs. 2–4, respectively. Part (a) of each figure shows the root-mean-square deviation of the music sequence, $\sigma(\Delta x)$; and part (b) is the power spectrum of the sequence. The sharp rise or drop in the values of $\sigma(\Delta x)$ near the end of the sequence reflects the finite-number-of-data effect and should be excluded from analysis. The solid lines superimposed on the graphs are obtained from least-square curve fitting of the data points in the appropriate intervals, and whose slopes represent respectively the Hurst exponent H of the graph in (a), and exponent β of the power spectrum in (b).

From music sequence I (Fig. 2(a)), for example, two different scaling regions can be clearly identified in the graph. Region with short-time scale designates that the interval between musical notes is no more than a few bars, while large time scale means musical notes are tens or even hundreds of bars away. The values of the

Table 2
Values of the Hurst exponent H and exponent β of the power spectrum obtained from various music-walk sequences (large time-interval analysis)

Large time-interval analysis	I	II	III[a]	IV(a)	IV(b)	V(a)	V(b)	VI(a)	VI(b)	VII(a)	VII(b)
H	0.0175	0.0005	0.22	0.0050	0.0075	0.0009	0.0082	0.0013	0.0069	0.0043	0.0020
β	0.93	1.08	1.43	1.01	0.92	1.09	0.91	1.17	1.09	0.91	1.10
$2H+1$	1.035	1.001	1.44	1.01	1.015	1.0018	1.0164	1.0026	1.0138	1.0086	1.004
Deviation	10.14%	7.89%	0.69%	0.00%	9.36%	9.03%	10.47%	16.70%	7.52%	9.78%	9.56%

[a]Music sequence III has only one scaling region; hence the data listed here for this particular sequence are the same as that from the short time-interval analysis listed in Table 1.

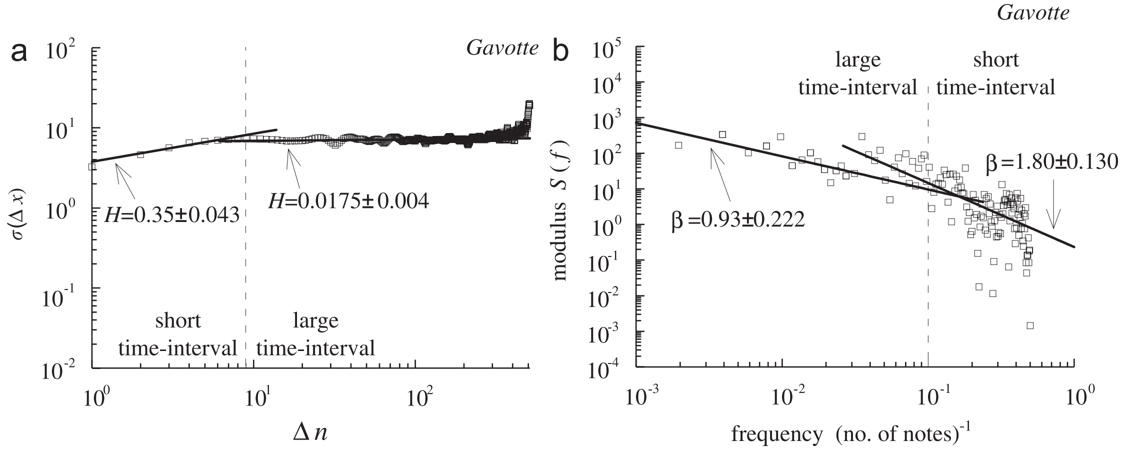

Fig. 2. (a) Root-mean-square deviation diagram and the associated Hurst exponent for the music-walk sequence *Gavotte*; (b) power spectrum diagram and the associated exponent β for the music-walk sequence *Gavotte*.

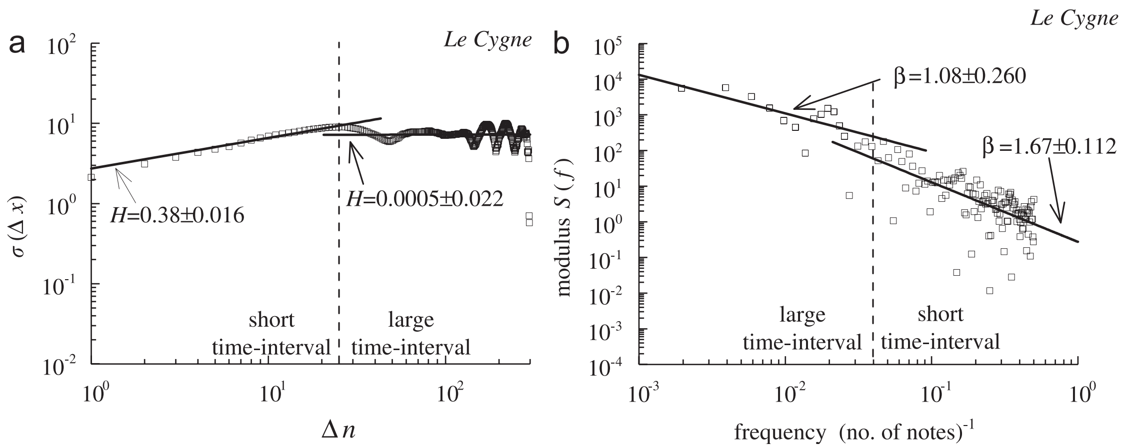

Fig. 3. (a) Root-mean-square deviation diagram and the associated Hurst exponent for the music-walk sequence *Le Cygne*; (b) power spectrum diagram and the associated exponent β for the music-walk sequence *Le Cygne*.

Hurst exponent for the graph in these two sections are 0.35 and 0.0175, respectively. The Hurst exponent is not equal to 0.5, implying a certain correlation between 'increments' of successive data values in the sequence. For music sequences, these increments represent the difference in pitch between adjacent notes, which is commonly referred to as the 'melody' of the music. Thus, it is reasonable to assert that music piece I exhibits

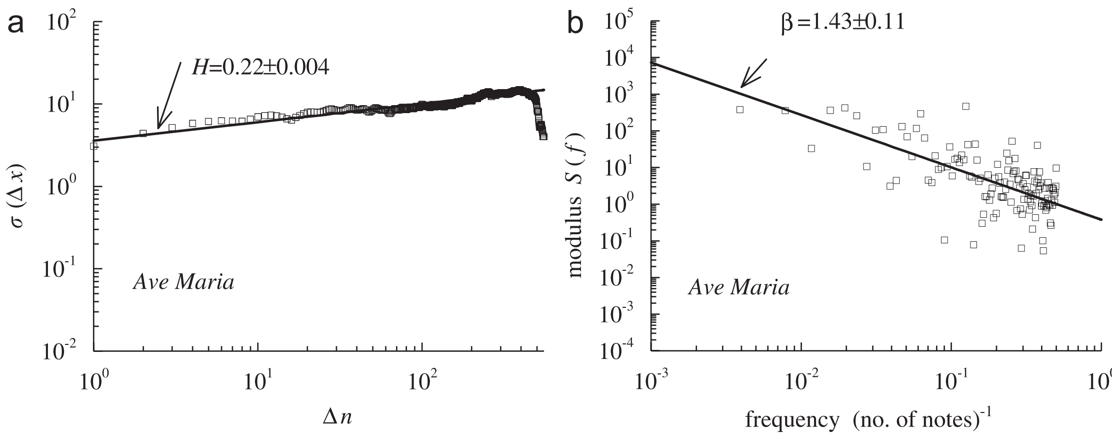

Fig. 4. (a) Root-mean-square deviation diagram and the associated Hurst exponent for the music-walk sequence *Ave Maria*; (b) power spectrum diagram and the associated exponent β for the music-walk sequence *Ave Maria*.

an anti-persistent correlation in melody in both short-scale and large-scale measures. On the other hand, the value of the Hurst exponent in the short-scale range (0.35) is somewhat closer to 0.5, indicating that the music sequence still preserves some degree of randomness in melody (changes of pitches) over an interval of a bar or two. Several bars later, the Hurst exponent drops to a value near 0, meaning that the melody of the music sequence exhibits a nearly perfect long-range correlation over the entire movement. That is, the melodic motion of the music piece at one time is still correlated with that in the long past before. This might perhaps reflect the underlying structure of all music—being full of variation in short measures while still preserving coherency in the long run. Our present analysis clearly discloses this genuine feature of music.

From Fig. 3(a) it can be observed that the H value (0.38) of music II in the short-time scale region is relatively larger than that of music I. As the Hurst exponent H and the fractal dimension D of the curve are related by equation $D = 2 - H$, a larger Hurst exponent implies a smoother music sequence, $x(n)$, and less obvious fluctuation in pitch over the examined time period. This can be proven by inspecting Fig. 1 where the curve of music II appears smoother locally than that of the previous one. As in music I, there are two different scaling regions in music II. The Hurst exponent corresponding to the large-time scale is 0.0005 (see the right part of Fig. 3(a)), showing again a nearly perfect long-range correlation in the melodic motion. Note in this diagram that the mild oscillations at the end of the curve indicate repetitions of measures in this musical piece, suggesting a periodic feature in the structure of the music sequence. Admittedly, it has already been observed from Fig. 1 that music II is more cyclic than music I.

Unlike previous two pieces of music, music sequence III shows only one scaling region (see Fig. 4(a)), and its Hurst exponent value is about 0.22, the smallest among three. That music III has the smallest Hurst exponent among the three works implies its sequence $x(n)$ is more rugged in appearance, with more noticeable fluctuations in pitch. Furthermore, the Hurst exponent 0.22 is further away from the value 0.5, indicating that music sequence III is somewhat 'less random' in pitch changes (i.e., shows greater correlation in the increments of adjacent values in the sequence), and hence lacks of diversity in melodic motion than the aforementioned two music sequences I and II. In fact, among the three musical pieces, music III sounds relatively monotonous. Therefore, the melody of a musical work can indeed be quantified and described by calculating the Hurst exponent of its score.

The power spectra of these three musical pieces are given in part (b) of Figs. 2–4. Basically, all energy spectra show the power-law $1/f^{\beta}$ tendency with $\beta > 0$. The gradual increase of the spectral energy toward low-frequency end of the spectrum is caused by the long-tailedness effect (slowly decaying autocorrelation function) of the music sequence—a typical behavior of a scaling noise with long-range correlation. Closer inspection suggests that both music sequences I and II also display two different scaling regions in their spectra, similar to that exhibited by the $\sigma(\Delta x)$ curve. The dividing frequency of these two regions can be estimated by locating the turning point of the $\sigma(\Delta x)$ curve (the location $(\Delta n)_c$ where the $\sigma(\Delta x)$ curve changes its slope); then the dividing frequency is roughly $1/(\Delta n)_c$. The right side of the $\sigma(\Delta x)$ diagram represents region with large time scale and hence corresponds to the low frequency part of the spectrum diagram; likewise, the left side of the $\sigma(\Delta x)$ diagram is associated with the high frequency part of the spectrum diagram. Either way, the β value calculated from both high and low frequency parts of the spectrum approximately satisfies the equation $\beta = 2H + 1$ (see Tables 1 and 2), indicating that music sequence indeed possesses the properties of an fBm.

The rest of the music pieces IV(a,b)–VII(a,b) studied in this paper are all duos. Results are summarized in Table 1 for short time-interval (high-frequency) analysis and Table 2 for large time-interval (low-frequency) analysis. It can be ascertained from these tables that the corresponding H and β values of the two musical parts in a duet are very close. The result suggests that as composers compile the two music parts in a duet, eventually a similar fractal structure and dimension would be formed. The theoretical reason for this is not yet clear. The results here may be coincidental and analyses of more samples are required for validation. Additionally, from Tables 1 and 2 it can be observed that both the H value and the β value satisfy the relation $\beta = 2H + 1$ well, which is demonstrated by the minor deviation percentages shown in the last rows of the tables.

Noticeably, the Hurst exponent values of all music-walk sequences analyzed in this study are smaller than 0.5 (an anti-persistent trend in melodic motion), and this may be a typical characteristic of music in general. It is doubtful whether a music score with a Hurst exponent far greater than 0.5 exists at all, because a Hurst exponent over 0.5 indicates persistency in the melodic motion of the music; that is, the tones of adjacent musical notes are well too positively correlated, making the music lacking in ‘surprises’ and sound boring.

The fact that the spectrum exponent β of each music sequence ranges between 0 and 2 reveals that music is neither unrelated random noise nor Brownian noise, with the strongest correlation, but is something else in-between, which integrates randomness and orderliness in tones arrangement. This echoes the properties of an anti-persistent fBm. In addition, it can be found that almost all musical works analyzed in this paper exhibit two different scaling regions, implying that same piece of music would differ in correlation and characteristic under different time scales. This is quite understandable. We have all possibly experienced a certain piece of music in which several measures demonstrate considerable variation while the whole movement has a consistent musical style.

Examining the β values for all music pieces studied here, it can be seen that the results are remarkably consistent, $1.4 < \beta < 1.8$ for high-frequency area of the spectra. The values are somewhat closer to the Brownian-noise value ($\beta = 2$), which corroborates the results of Boon and Decroly [30]. This observation also confirms the qualitative results of Nettheim based on the analysis of five melodic lines (Bach, Mozart, Beethoven, Schubert and Chopin) [31]. Nevertheless, in the low-frequency area, the β values eventually all turn out to be close to 1, that of a typical $1/f$-noise having perfect long-range correlation.

Finally, it has been mentioned before that two pitches are in harmony if they have a ratio of small integers. Therefore, when two tones are being played simultaneously, it is the pitch difference between them that determines whether or not they are in harmony—for example, the combination of two tones with pitch interval $i = 7$ (a perfect fifth) is sonorous, while a difference of pitch interval $i = 6$ (a diminished fifth) is dissonant. Therefore, the sequences of pitch differences obtained by deduction between the two musical parts in the works of Beethoven, Mozart, Pachelbel, and Bach (music sequences IV(a,b)–VII(a,b)) are processed again with the above Hurst exponent and power spectral analyses. The results from the Beethoven’s work (music IV(a,b)) are shown in Fig. 5. Since the sequence derived from the pitch differences between two musical parts also displays the feature of power-law scaling and satisfies the equation $\beta = 2H + 1$, it can be deduced that the progression of harmonic in a duet is also of a fractal nature. Values of respective Hurst exponent H and frequency spectrum exponent β are given in Tables 3 and 4.

5. Conclusion

The fractal nature of music has been constructed by applying the Hurst exponent and Fourier spectral analyses to the music-walk sequences converted from various music scores. Both the root-mean-square deviation and the power spectrum of the music-walk sequences exhibit a power-law scaling property, a

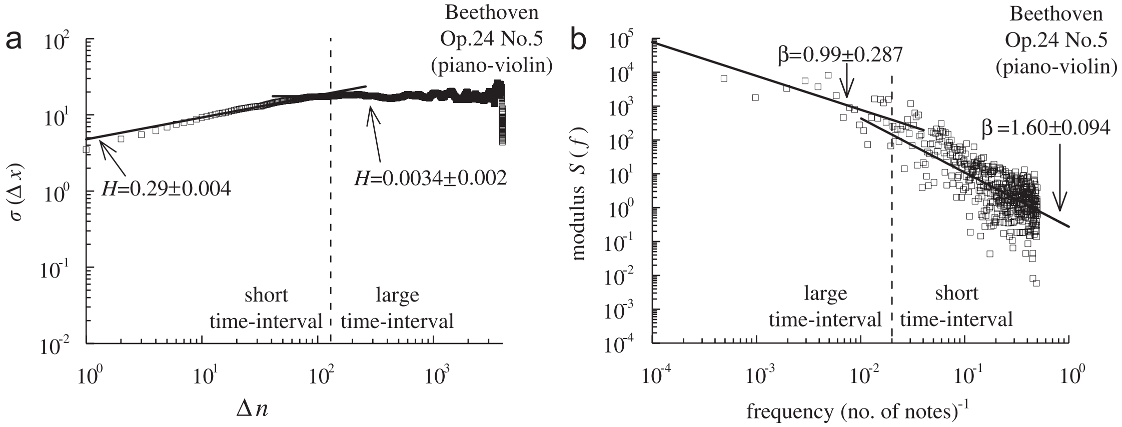

Fig. 5. (a) Root-mean-square deviation diagram and the associated Hurst exponent for the sequence deduced from the pitch differences between two musical parts of Beethoven's work Op. 24 No. 5; (b) power spectrum diagram and the associated exponent β for the sequence deduced from the pitch differences between two musical parts of Beethoven's work Op. 24 No. 5.

Table 3
Values of the Hurst exponent H and exponent β of the power spectrum for the sequence deduced from the pitch differences between two musical parts of various duets (short time-interval analysis)

Short time-interval analysis	IV(a)–IV(b)	V(a)–V(b)	VI(a)–VI(b)	VII(a)–VII(b)
H	0.29	0.23	0.33	0.14
β	1.60	1.44	1.51	1.26
$2H+1$	1.58	1.46	1.66	1.28
Deviation	1.27%	1.37%	9.04%	1.56%

Table 4
Values of the Hurst exponent H and exponent β of the power spectrum for the sequence deduced from the pitch differences between two musical parts of various duets (large time-interval analysis)

Large time-interval analysis	IV(a)–IV(b)	V(a)–V(b)	VI(a)–VI(b)	VII(a)–VII(b)
H	0.0034	0.0029	0.0102	0.0152
β	0.99	1.04	1.15	1.03
$2H+1$	1.0068	1.0058	1.0204	1.0304
Deviation	1.67%	3.40%	12.70%	0.04%

resemblance to the long-range correlation found in many naturally occurring fluctuating phenomena. Music is self-similar, both in audition (by listening to the successive changes in the acoustic frequency) and in visual representation (by looking at the up-and-down fluctuation of notes on the staffs). The melodic arrangement of the entire movement of a music score is, to some extent, similar to the tone variation in just a few bars. The lack of a coherent fractal structure in a random fluctuation is one of the key points that essentially discern noise from music.

Closer investigation shows that music-walk sequences generally display a scaling property in two apparently different temporal scales with two different Hurst exponents. Short time-interval analysis gives typical Hurst exponent values $H = 0.2-0.4$; meaning that the tone change in music is normally anti-persistent in trend while at the same time still preserving certain degree of randomness in its progression over a few measures. Large time-interval analysis gives $H \approx 0$, which results in a near $1/f$-type power spectrum, signifying a perfect long-range correlation of melodic motion over the entire musical movement. This may reflect the basic idea about music—tone variation in short measures makes the music abundant in expression, while long-term correlation connects hundreds of bars of different melodious messages into a coherent piece of music. The fact the exponents of the power spectra of all music-walk sequences studied here range between 0 and 2 once again makes evident that music integrates both randomness and certainty like many other natural phenomena do.

It is also found that the Hurst exponent values of the musical works analyzed in this study are all smaller than 0.5, indicating that music can be commonly treated as an anti-persistent fBm. This may be a typical characteristic of music in general. Because a Hurst exponent greater than 0.5 implies the persistency of tone changes in music, one part of music would be too similar to other parts in melodic motion, making the music lacking in variability and sound monotonous.

The way in which music communicates with the mind may well not be realized in entirety until we achieve a thorough understanding of how human's mentality, sense and sensitivity are commanded by the brain. We do, however, begin to gain knowledge about the universality of the underlying structures embedded in diverse works of nature. If music is invented to mimic a certain harmony in nature, then mountains are songs; rivers are lyrics. Modern musicians use the concepts of fractals and chaos in their compositions [32–37]; music composed in this way must of course be part of the fractal scene.

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